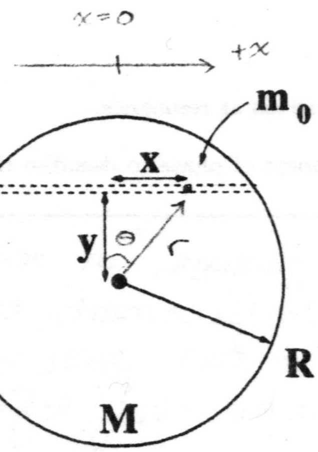


Inside a spherically-symmetric distribution of gravitational charge (mass), at at distance r from the center of the distribution, the gravitational field has a magnitude $g(r) = \frac{G\,m(r)}{r^2}$, where $m(r)$ is the amount of charge (mass) contained in a concentric sphere of radius r. The field itself points radially-inward, towards the center of the distribution.

- 1a) (5 points) Consider a uniform spherical distribution of mass M, radius R. Find $m(r)$ and $g(r)$ for all $r \leq R$.

uniform:

$\rho = \rho$

$$\frac{3M}{4\pi R^3} = \frac{3m(r)}{4\pi r^3} \Rightarrow$$

$$\boxed{m(r) = M \frac{r^3}{R^3} \qquad g(r) = \frac{GMr}{R^3}}$$

- 1b) (15 points) Now suppose a thin tunnel is drilled through the sphere, such that, at closest approach, an object in the tunnel would be a distance y from the center of the distribution (as shown). Show that if a point-charge (mass) m_0 is placed in the tunnel a distance x from that point of closest approach and released, it will execute simple-harmonic motion about the point of closest approach. Find the angular frequency associated with that motion.

$$\vec{F_g} = m_0 \vec{g}(\vec{r})$$

$$\vec{F_g} = -\frac{GMm_0 r}{R^3}\hat{r}$$

$$F_{g,x} = -\frac{GMm_0 r \sin\theta}{R^3}$$

but $x = r\sin\theta$, so

$$F_{g,x} = -\frac{GMm_0}{R^3}x$$

linear restoring force :)

Within the tunnel, gravity exerts a linear restoring force on m_0 of effective force constant $k = \frac{GMm_0}{R^3}$

Since $\omega_0 = \sqrt{k/m}$

$$\boxed{\omega_0 = \sqrt{\frac{GM}{R^3}}}$$

- 1c) (10 points) Suppose the sphere is the planet Earth (neglect the fact that it is rotating) and the tunnel is used to deliver mail from one town to another. How long will it take for mail deposited (at rest) on one side of the tunnel to arrive on the other ($R_{earth} \approx 6.4 \times 10^6$ m, and - no - you don't get to use a calculator)? How would this length of time compare to the time required for mail, similarly deposited, to travel the length of a tunnel through the center of the Earth? Explain.

$$T = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{R_E^2}{GM_E} R_E} = 2\pi\sqrt{\frac{R_E}{g_E}}$$

$$T \simeq 6.3\sqrt{\frac{64\times10^5\,m}{10\,m/s^2}}$$

$$T \simeq 6.3\,(800\,s) = 5040\,s \simeq 84\,min$$

← interestingly enough, roughly the period of a satellite in low Earth orbit!

⇒ It will only take half a period to travel from one end to the other, so...

$$\boxed{\Delta t \simeq 42\,min}$$

Since T doesn't depend on y, the elapsed time, end-to-end on <u>any</u> tunnel should be (ideally) ~ 42 min

The motion of a driven oscillator is described by the differential equation

$$\frac{d^2x}{dt^2} + 2\beta\,\frac{dx}{dt} + \omega_0^2\,x = a\,\cos(\Omega t)$$

Compare this to:

$$\frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \omega_0^2 x = \frac{F_0}{m}\cos(\Omega t)$$

$\beta = \frac{b}{2m}$

$a = F_0/m$

- 2a) (10 points) Write the equation of motion ($x(t)$) in as much detail as you can. The more correct detail you supply, the more points you will receive.

$$x(t) = A_0 e^{-\beta t}\cos(\omega t + \phi) + A(\Omega)\cos(\Omega t + \xi)$$

$$\omega = \sqrt{\omega_0^2 - \beta^2}$$

$$A(\Omega) = \frac{a}{\sqrt{(\omega_0^2 - \Omega^2)^2 + (2\beta\Omega)^2}}$$

$$\tan\xi = \frac{-2b\Omega}{\omega_0^2 - \Omega^2}$$

- 2b) (5 points) What angular frequency will the system oscillate at if $a = 0$?

→ damped, undriven oscillator

$$\boxed{\omega = \sqrt{\omega_0^2 - \beta^2}}$$

- 2c) (5 points) What (angular) driving frequency will maximize the amplitude of the system after it has reached steady-state?

$$\boxed{\Omega_{res} = \sqrt{\omega_0^2 - 2\beta^2}}$$

- 2d) (10 points) Find the maximum amplitude attainable by the system in steady-state and show that it has a simple relationship to the quantity you calculated in part *b*.

$$A(\Omega_{res}) = \frac{a}{\sqrt{(\omega_0^2 - \omega_0^2 + 2\beta^2)^2 + 4\beta^2(\omega_0^2 - 2\beta^2)}}$$

$$A(\Omega_{res}) = \frac{a}{\sqrt{4\beta^4 + 4\beta^2\omega_0^2 - 8\beta^4}}$$

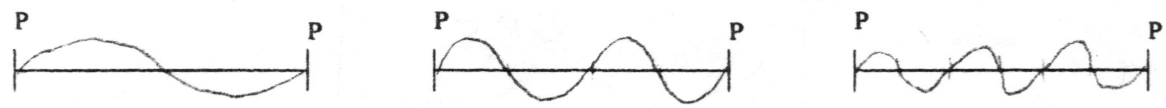
$$\boxed{\begin{aligned} A(\Omega_{res}) &= \frac{a}{2\beta\sqrt{\omega_0^2 - \beta^2}} \\ A(\Omega_{res}) &= \frac{a}{2\beta\omega_{damp}} \end{aligned}}$$

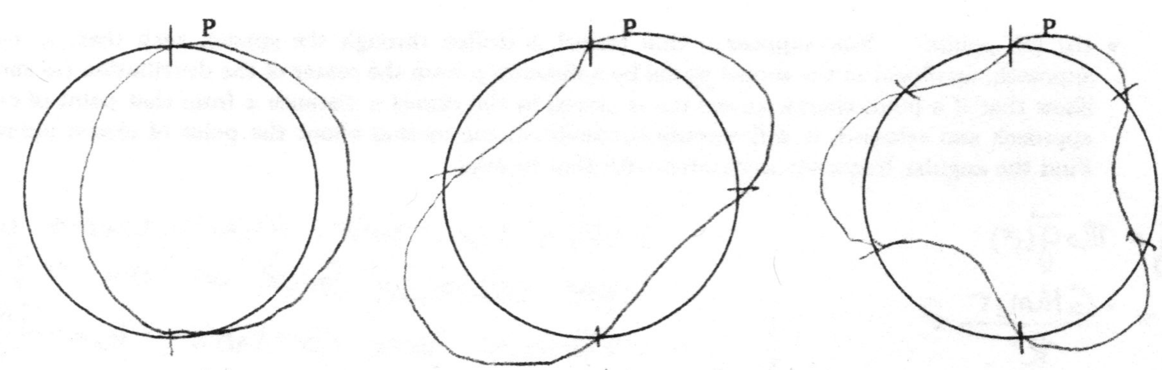
3) Consider a champagne glass driven at resonance...

- 3a) (5 points) Use the concept of phase to describe the boundary condition for resonant waves on the rim of the glass.

→ There are no reflections... in order to get constructive reinforcement, the waves must return to each point in the glass with the phase they left with

- 3b) (15 points) Use the templates below to sketch the first three resonant modes on the rim of a champagne glass (hint: it might be easier if you sketch the sine waves on the horizontal lines first). In each case, describe how the motion would look to an external observer.





The glass would appear to be shaking back and forth

The glass would appear to be alternately stretched and squished along orthogonal lines.

Chances are, this mode is harder to see - lots of bending and flexing along the rim

- 3c) (5 points) **If sound moves through the glass at a rate V_{snd} and the glass has a diameter D, what frequencies will cause the glass to vibrate with large amplitude displacements?**

$$N\lambda = 2\pi R = \pi D$$

$$N\frac{V_{snd}}{f} = \pi D$$

$$\boxed{f_N = N\frac{V_{snd}}{\pi D}}$$

- 3d) (5 points) **Would the fundamental frequency or a harmonic be more effective at shattering the glass? Discuss.**

The higher harmonics place greater stress and strain on the rim (lots of bending and flexing) - If I could generate sufficient energy to rattle the glass at a higher harmonic, I would probably try that

But it's usually easier to produce very energetic driving waves at lower frequencies - In practice, we usually use the fundamental