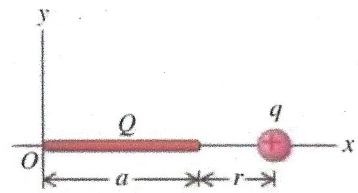
1B-1 Fall 2020: Quiz 4A

Show all your work and use proper units throughout. This quiz is open-book but not open-Chegg and must be completed without help. Please write your answers into the boxes. If you submit your work with your own formatting please try to submit the same number of pages as the template (5).

1. A positive point charge q is located on the x-axis as shown in the figure. A positive charge Q is distributed uniformly along the x-axis from $x = 0$ to $x = a$. The distance between q and the right end of Q is r.



a) Calculate the x- and y-components of the electric field produced by the charge distribution Q on the points on the positive x-axis where $x > a$. *[5 points]*

The field caused by a piece of charge dq at the point $(x', 0)$, evaluated at the point $(x,0)$ is $\frac{dq}{4\pi\varepsilon_0(x-x')^2}\hat{i}$. So the total electric field at $(x,0)$ is $\int \frac{dq}{4\pi\varepsilon_0(x-x')^2}\hat{i}$ with the appropriate limits. We have $dq = \frac{Q}{a}dx'$

So $$\vec{E}(x,0) = \int_{x'=0}^{a} \frac{\frac{Q}{a}dx'}{4\pi\varepsilon_0(x-x')^2}\hat{i} = \frac{Q}{4\pi\varepsilon_0 a}\hat{i}\left[-\frac{1}{x'-x}\right]_{x'=0}^{a}$$

$$= \frac{Q}{4\pi\varepsilon_0 a}\hat{i}\left(-\frac{1}{a-x} + \frac{1}{0-x}\right) = \frac{Q}{4\pi\varepsilon_0 a}\frac{x-(x-a)}{(x-a)x}\hat{i}$$

$$= \boxed{\frac{Q}{4\pi\varepsilon_0}\frac{1}{(x-a)x}\hat{i}}.$$

(In terms of $r = x-a$, this is $\vec{E} = \frac{Q}{4\pi\varepsilon_0}\frac{1}{r(r+a)}\hat{i}$.)

b) Calculate the vector force that the charge distribution Q exerts on q. *[5 points]*

$$\vec{F} = q\,\vec{E}(x,0) = \boxed{\frac{qQ}{4\pi\varepsilon_0}\frac{1}{(x-a)x}\,\hat{i}}\,.$$

(In terms of $r = x - a$ this is $\vec{F} = \frac{qQ}{4\pi\varepsilon_0}\frac{1}{r(r+a)}\hat{i}$.)

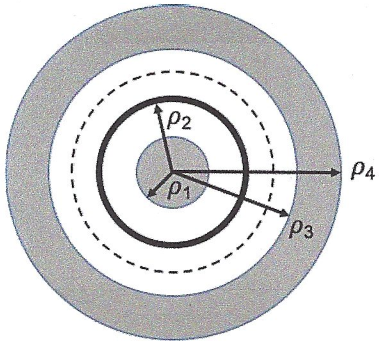
c) Show that if r> a, the magnitude of the force in part (b) is approximately $\frac{Qq}{4\pi\epsilon_0 r^2}$. *[5 points]*

If $r \gg a$ then $\frac{a}{r} \ll 1$ so we can think of making an approximation by keeping just the first nonzero term in a Taylor expansion in terms of the small parameter $\frac{a}{r}$.

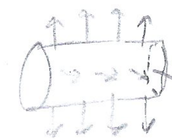
$$\vec{F} = \frac{qQ}{4\pi\varepsilon_0} \frac{1}{r(r+a)} \hat{i} = \frac{qQ}{4\pi\varepsilon_0} \frac{1}{r^2} \frac{1}{1+\frac{a}{r}} \hat{i} = \frac{qQ}{4\pi\varepsilon_0} \frac{1}{r^2}\Big(1 \underbrace{- \frac{a}{r} + \left(\frac{a}{r}\right)^2 - \ldots}_{\text{small}}\Big)\hat{i}$$

$$\approx \frac{qQ}{4\pi\varepsilon_0} \frac{1}{r^2} \hat{i}.$$

2. Consider the infinitely long, cylindrically symmetric charge configuration shown below. The inner most cylinder is composed of an insulator of radius of ρ_1 and has charge homogeneously distributed throughout. The linear charge density is ρ_L. This cylinder is surrounded by air, which is surrounded by an infinitely thin cylindrical conductor of radius ρ_2 and linear charge density $2\rho_L$, which is surrounded by air, which is surrounded by an insulating cylindrical shell of inner radius ρ_3 and outer radius ρ_4. The linear charge density of the outer shell is -ρ_L, with the charge homogeneously distributed throughout.



a) Derive an expression for the vector electric field as a function of radius ρ everywhere inside the outermost cylindrical shell ($\rho_3 < \rho < \rho_4$). *[8 points]*



Consider the electric flux through a cylinder of length L and radius ρ where $\rho_3 < \rho < \rho_4$. The electric field is radial away from the center of the configuration, so the flux out the ends of the cylinder is zero and the flux out the round side is $E(\rho)\, L\, 2\pi\rho$. By Gauss's law $E(\rho) L 2\pi\rho = \frac{Q_{enclosed}}{\varepsilon_0}$. There are three contributions to $Q_{enclosed}$: $L\rho_L$ for the center insulator, $L 2\rho_L$ for the cylindrical conductor at $\rho = \rho_2$, and a contribution for the part of the outer shell that is contained within the surface. We are not given the volume charge density of the outer shell, only its linear charge density $-\rho_L$. But we can find its volume charge density: $\frac{\text{charge}}{\text{volume}} = \frac{-L\rho_L}{L(\pi\rho_4^2 - \pi\rho_3^2)} = \frac{-\rho_L}{\pi(\rho_4^2 - \rho_3^2)}$.

So the charge in the outer shell that lies within the surface is $\frac{-\rho_L}{\pi(\rho_4^2-\rho_3^2)} L(\pi\rho^2 - \pi\rho_3^2) = -L\rho_L \frac{\rho^2-\rho_3^2}{\rho_4^2-\rho_3^2}$.

Thus

$$E(\rho) = \frac{Q_{enclosed}}{L 2\pi\rho\,\varepsilon_0} = \frac{L\rho_L + L2\rho_L - L\rho_L \frac{\rho^2-\rho_3^2}{\rho_4^2-\rho_3^2}}{L 2\pi\rho\varepsilon_0} = \boxed{\frac{3\rho_L - \rho_L \frac{\rho^2-\rho_3^2}{\rho_4^2-\rho_3^2}}{2\pi\varepsilon_0\rho}}.$$

The direction is $\boxed{\hat{\rho} = \vec{a}_\rho = \text{"radially outward."}}$

b) What is the electric flux through the cylindrical surface shown by the dashed line at $\rho_2 < r < \rho_3$? Explain your reasoning. *[2 points]*

If you interpret the surface as a cylinder of length L then the electric flux, by Gauss's law is $\frac{Q_{enclosed}}{\varepsilon_0} = \frac{L\rho_L + L2\rho_L}{\varepsilon_0} = \frac{3L\rho_L}{\varepsilon_0}$.

If you interpret the surface as an infinite cylinder the flux is infinite.

c) Qualitatively correct sketch the magnitude of the electric field as a function of radius from $\rho = 0$ to $\rho > \rho_4$. *[5 points]*