Physics 1B - Midterm 1

Due Tuesday, April 21 at 11:59 PM on Gradescope

UCLA / Spring 2020 / Brian Naranjo

Problems

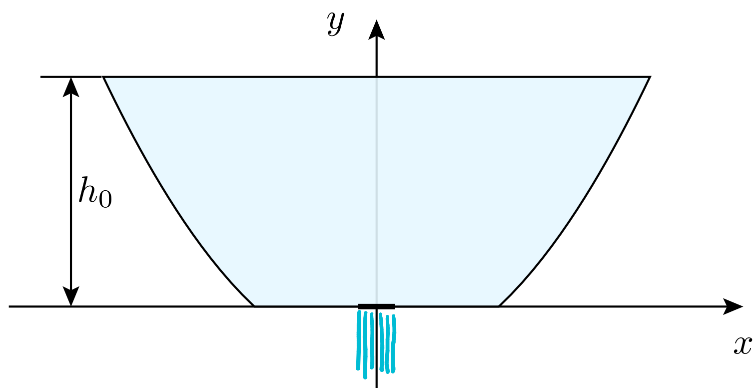
1) Fluid mechanics

Taking the y axis to point upward, consider an upright vase of height h_0 whose base is centered on the origin. The vase is rotationally symmetric about the y axis with radial profile

$$r(y) = \alpha\sqrt{y + y_0},$$

where α and y_0 are positive constants.

The vase's base is flat with radius $r(y = 0) = \alpha\sqrt{y_0}$, and the vase's rim has radius $r(y = h_0) = \alpha\sqrt{h_0 + y_0}$. Initially, the vase is filled to the rim with a fluid of density ρ. The fluid at the top of the vase is exposed to the atmosphere of pressure p_0. The vase has a small circular drain of radius r_0 (assume $r_0 << \alpha\sqrt{y_0}$ so that the drain is much smaller than the base) at the origin, through which the fluid flows out to the atmosphere, as shown,



In this problem, we will use Bernoulli's principle and the equation of continuity to calculate the time Δt that it takes for all of the fluid to flow out.

a) What is the initial total fluid volume V in the vase?

b) In applying Bernoulli's principle to the fluid's top surface and to the vase's drain, we would like to neglect the kinetic energy term evaluated at the fluid's top surface because it is much smaller than the kinetic energy term evaluated at the vase's drain. Confirm this by calculating

$$\frac{[(1/2)\rho v^2]_{\text{top}}}{[(1/2)\rho v^2]_{\text{drain}}},$$

and showing that it is a small value.

Then use Bernoulli's principle to find the initial *volume flow rate* Q (dimensions of volume per time) out of the drain. If we were able to maintain a volume flow rate at this constant value, how long would it take to drain the vase?

c) At time t, the fluid level is at height $y(t)$. Derive a relation of the form

$$\frac{dy}{dt} = -f(y).$$

As previously, in your application of Bernoulli's principle, you may drop the kinetic energy term of the fluid's top surface.

d) The previous differential equation is separable, and may be integrated as

$$\int_0^{\Delta t} dt = \int_0^{h_0} \frac{1}{f(y)}\, dy.$$

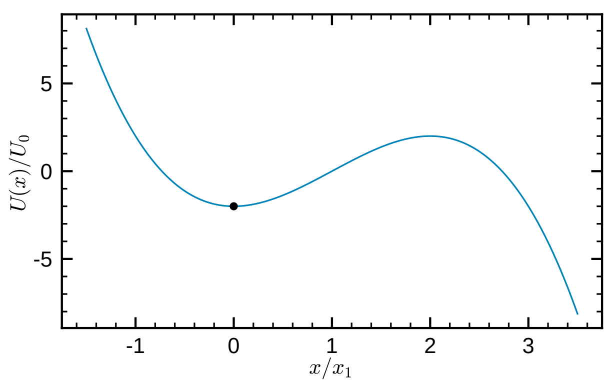
Carry this out to find the total time Δt that it takes for all of the fluid to flow out.

2) Periodic motion

A particle of mass m has potential energy

$$U(x) = U_0 \left[\frac{3(x - x_0)}{x_0} - \frac{(x - x_1)^3}{x_1^3} \right],$$

where U_0 and x_1 are positive constants, and x_0 is to be determined.



a) Calculate $U'(x)$ and $U''(x)$. What are the conditions for there to be stable equilibrium at $x = 0$? Find x_0 so that both of these conditions are satisfied.

b) Use Taylor's theorem to express $U(x)$,

$$U(x) = U(0) + U'(0)x + \frac{1}{2}U''(0)x^2 + \cdots$$

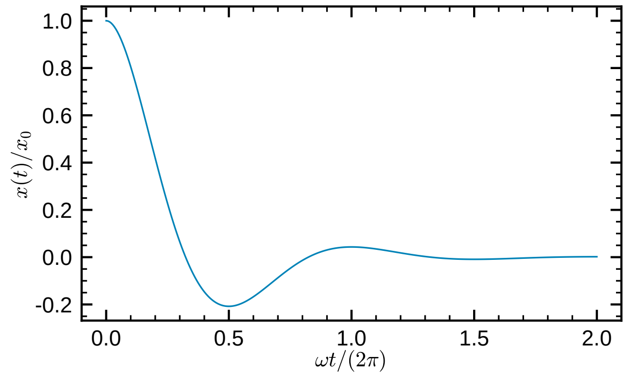
up to, and including, the quadratic term. Then, identify the spring constant k for small oscillations about the origin.

c) Assuming that the particle feels a viscous damping force of the form $F = -bv$, write down the particle's equation of motion expressed in terms of $\beta \equiv b/(2m)$ and $\omega_0^2 \equiv k/m$.

d) We assume that $\omega_0 > \beta$, so that the motion is underdamped. Then, a general solution may be written

$$x(t) = Ae^{-\beta t}\cos(\omega t + \phi),$$

where $\omega \equiv \sqrt{\omega_0^2 - \beta^2}$. We are **not** making the light damping approximation in this problem, so that the motion may be heavily damped but still oscillatory. If the initial conditions are $x(0) = x_0$ and $x'(0) = 0$, then solve for A and $\tan\phi$. (Hint: $A \neq x_0$). Possible motion is



3) Mechanical waves

Transverse waves on a string are described by a wave function $y(x,t)$ that consists of sinusoidal left-traveling and right-traveling waves of the same amplitude A, angular frequency ω, and wave number k,

$$y(x,t) = A\cos(kx - \omega t) + A\cos(kx + \omega t + \phi)$$

a) If the string has mass density μ, then what is the string's tension T?

b) If there is a **free** boundary condition at $x = 0$, then what restriction must $y(x,t)$ satisfy at $x = 0$? What value of ϕ ensures that this restriction is satisfied? Using this value of ϕ and the trig identity

$$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2},$$

factor the wave function into separate spatial and temporal factors, e.g.,

$$y(x,t) = f(x)g(t).$$

c) In addition to the **free** boundary condition at $x = 0$, further assume that there is a **free** boundary condition at $x = L$. Only certain values of wave-vector k are now allowed, which we label k_n for $n = 1, 2, 3, \ldots$ (here, we only take positive values of k_n). Use the boundary condition at $x = L$ to determine k_n. Then, use $v = \omega/k$ to find ω_n. What are k_n and ω_n?

d) Recall that a standing wave transmits no net power. However, the string definitely does have energy – it just stays in one place. The wave function $y_n(x,t)$ of the nth harmonic is obtained by substituting k_n and ω_n back into your expression for $y(x,t)$ from the end of part b. Calculate the *instantaneous* kinetic energy of the nth harmonic,

$$K_n(t) = \int_0^L \frac{1}{2}(\mu dx)\left(\frac{\partial y_n}{\partial t}\right)^2.$$

Solutions

1) Fluid mechanics

a)

$$V = \int_{\text{vase}} dV = \int_0^{h_0} \pi\alpha^2(y+y_0)\,dy = \pi\alpha^2\left[\frac{y^2}{2} + y_0 y\right]_0^{h_0} = \pi\alpha^2 h_0\left(\frac{h_0}{2} + y_0\right)$$

b) The equation of continuity says that the quantity $Q = Av$ is conserved along streamtubes,

$$\frac{[(1/2)\rho v^2]_{\text{top}}}{[(1/2)\rho v^2]_{\text{drain}}} = \frac{[v^2]_{\text{top}}}{[v^2]_{\text{drain}}} = \frac{[Q/A^2]_{\text{top}}}{[Q/A^2]_{\text{drain}}} = \frac{r_0^4}{\alpha^4(h_0+y_0)^2}.$$

This ratio is much less than one because we have assumed $r_0 \ll \alpha\sqrt{y_0} < \alpha\sqrt{h_0+y_0}$. Bernoulli's principle applied at the top and at the drain gives

$$p_0 + \rho g h_0 + (1/2)\rho v_{\text{top}}^2 = p_0 + (1/2)\rho v_{\text{drain}}^2.$$

Dropping the top kinetic energy term, which we have now justified, gives

$$v_{\text{drain}} = \sqrt{2gh_0}, \quad \text{and} \quad Q_0 = [Av]_{\text{drain}} = \pi r_0^2\sqrt{2gh_0}$$

Finally,

$$\Delta t_0 = \frac{V}{Q_0} = \frac{\alpha^2}{r_0^2}\sqrt{\frac{h_0}{2g}}\left(\frac{h_0}{2} + y_0\right)$$

c) Apply Bernoulli's principle just like we did previously, except now the top of the fluid is at height y instead of h_0,

$$Q(y) = \pi r_0^2\sqrt{2gy}.$$

The volume flow rate gives the rate at which the fluid volume in the vase is decreasing,

$$Q(y) = -\frac{dV}{dt} = -\pi r^2(y)\frac{dy}{dt} = -\pi\alpha^2(y+y_0)\frac{dy}{dt},$$

so that

$$\frac{dy}{dt} = -\frac{r_0^2\sqrt{2gy}}{\alpha^2(y+y_0)} \equiv -f(y)$$

d)

$$\Delta t = \frac{\alpha^2}{r_0^2\sqrt{2g}}\int_0^{h_0}\frac{y+y_0}{\sqrt{y}}\,dy = \frac{\alpha^2}{r_0^2}\sqrt{\frac{h_0}{2g}}\left(\frac{2}{3}h_0 + 2y_0\right)$$

Note that $\Delta t > \Delta t_0$.

2) Periodic motion

a)

$$U'(x) = U_0 \left[\frac{3}{x_0} - \frac{3(x-x_1)^2}{x_1^3} \right]$$
$$U''(x) = -\frac{6U_0(x-x_1)}{x_1^3}$$

For stable equilibrium at the origin, we need

$$U'(0) = U_0 \left[\frac{3}{x_0} - \frac{3}{x_1} \right] = 0$$
$$U''(0) = \frac{6U_0}{x_1^2} > 0.$$

Take $x_0 = x_1$ to satisfy both these conditions.

b)

$$U(x) = -2U_0 + \frac{1}{2}\left(\frac{6U_0}{x_1^2}\right) x^2 + \cdots$$
$$k = \frac{6U_0}{x_1^2}$$

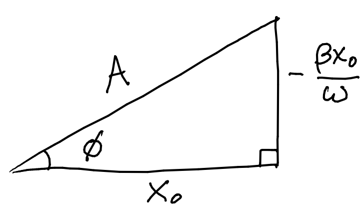
c)

$$x'' + 2\beta x' + \omega_0^2 x = 0$$

d) Don't assume light damping, so include both terms in velocity,

$$x'(t) = -\beta x(t) - \omega A e^{-\beta t} \sin(\omega t + \phi).$$

$$x(0) = x_0 = A\cos\phi \quad \Longrightarrow \quad \cos\phi = x_0/A$$
$$x'(0) = 0 = -\beta A\cos\phi - \omega A \sin\phi \quad \Longrightarrow \quad \sin\phi = -\beta x_0/(\omega A)$$



Therefore,

$$A = x_0\sqrt{1 + (\beta/\omega)^2}$$
$$\tan\phi = -\beta/\omega$$

3) Mechanical waves

a) For a string, $v = \sqrt{T/\mu}$, and phase velocity for a sinusoidal wave is always $v = \omega/k$,

$$T = \mu v^2 = \mu\omega^2/k^2$$

b) At a free boundary condition, the slope of the the transverse wave function must be zero,

$$\left.\frac{\partial y}{\partial x}\right|_{(0,t)} = 0.$$

This implies that $\phi = 0$. You can either show this by directly evaluating the slope at $x = 0$, or, you can use our result about how to satisfy a free boundary condition with left-traveling and right-traveling waves.
Applying the trig identity,

$$y(x,t) = 2A\cos(kx)\cos(\omega t)$$

c) Enforcing a free boundary condition at $x = L$, so that the slope at $x = L$ vanishes, restricts the allowed values of k and ω,

$$k_n = \frac{n\pi}{L} \text{ and } \omega_n = \frac{n\pi v}{L} \qquad n = 1, 2, 3, \ldots$$

d) The wave function for the nth harmonic is

$$y_n(x,t) = 2A\cos(k_n x)\cos(\omega_n t).$$

The string's transverse velocity is

$$\frac{\partial y_n}{\partial t}(x,t) = -2\omega_n A\cos(k_n x)\sin(\omega_n t)$$

At any instant time, we can find the nth harmonic's *instantaneous* kinetic energy by integrating over the string,

$$\begin{aligned} K_n(t) &= \int_0^L \frac{1}{2}(\mu dx)\left(\frac{\partial y_n}{\partial t}\right)^2 \\ &= 2\mu(\omega_n A)^2\sin^2(\omega_n t)\int_0^L \cos^2(k_n x)\,dx \\ &= \mu L(\omega_n A)^2\sin^2\omega_n t \end{aligned}$$

Nota bene: The two following integrals are equal because of the boundary conditions,

$$I \equiv \int_0^L \cos^2\left(\frac{n\pi x}{L}\right)\,dx = \int_0^L \sin^2\left(\frac{n\pi x}{L}\right)\,dx.$$

Then, using a trig identity,

$$2I = \int_0^L \left[\cos^2\left(\frac{n\pi x}{L}\right) + \sin^2\left(\frac{n\pi x}{L}\right)\right]\,dx = \int_0^L dx = L,$$

we have

$$I = L/2$$

These integrals appear a lot, so it is helpful to remember this.