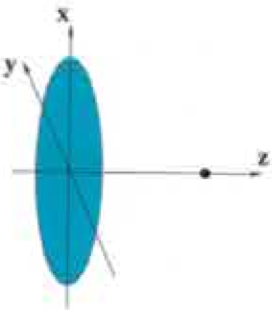


A thin, non-conducting disk of charge Q and radius R, centered on the origin, lies in the x, y-plane, as shown. Electric charge has been distributed over the disk such that the surface charge density,

$$\sigma(r) \propto (R^2 - r^2)$$

- 3a) (10 points) Normalize the surface charge density (that is, find the proportionality constant and write the charge density in its final, *normalized* form.

$$q_{in}(r) = \int dq = \int \sigma(r)\, dA = \int c\,(R^2 - r^2)\, dA$$

$$= \int_0^r c\,(R^2 - r^2)\, dA$$

$$= \int_0^r c\,(R^2 - r^2)\, \pi r^2\, dr$$

$$= \pi c \int_0^r (R^2 - r^2)\, r^2\, dr$$

$$= \pi c \int_0^r R^2 r^2 - r^4\, dr$$

$$= \pi c \left[\tfrac{1}{3} R^2 r^3 - \tfrac{1}{5} r^5 \right]_0^r$$

$$= \pi c \left(\tfrac{1}{3} R^2 r^3 - \tfrac{1}{5} r^5 \right)$$

$$Q = \pi c \left(\tfrac{1}{3} R^5 - \tfrac{1}{5} R^5 \right)$$

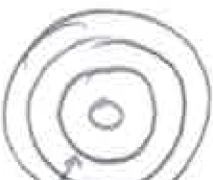
$$Q = \pi c \left(\left(\tfrac{5}{15} - \tfrac{3}{15} \right) R^5 \right)$$

$$Q = \pi c \left(\tfrac{2}{15} R^5 \right)$$

$$\boxed{c = \frac{Q}{\pi R^5}\frac{15}{2}} \quad = \quad \boxed{\sigma(r) = \frac{15 Q}{2\pi R^5}(R^2 - r^2)}$$

- 3b) (10 points) Find the (vector) electric field at a point z along the $+z$-axis. For full credit, use the normalized surface charge density. You *may* receive partial credit if you use c for the normalization constant, instead.

b/c of symmetry $\vec{E} = \vec{E}_z \hat{z}$



use rings to build disk

$$d\vec{E} = \frac{k\,dq\,z}{(r^2+z^2)^{3/2}}\hat{z} \qquad d\vec{E} = \frac{k\,dq}{r^2}\hat{r}$$

$$k = \frac{1}{4\pi\varepsilon_0}$$

$$dq = \sigma\, dA$$

$$dq = \frac{15\,Q}{\cancel{7}\cancel{\pi} R^5}(R^2-r^2)\cancel{2}\cancel{\pi} r\,dr$$

$$\int d\vec{E} = \hat{z}\int_0^R \frac{k\,15Q(R^2-r^2)\,r\,z}{R^5\,(r^2+z^2)^{3/2}}\,dr$$

$$\int d\vec{E} = \hat{z}\,\frac{kQ\,15z}{R^5}\int_0^R \frac{(R-r^2)\,r}{(r^2+z^2)^{3/2}}\,dr = \int \frac{(R-(u+z^2))}{2\,u^{3/2}}\,du$$

$u = r^2+z^2$

$du = 2r\,dr$

$$= \frac{1}{2}\left[\int \frac{R+z^2}{u^{3/2}} - \frac{u}{u^{3/2}}\,du\right]_{(0)}^{(R)} \qquad u^{-1/2} \quad u^{-3/2}$$

$$= \frac{1}{\cancel{2}}\left[\frac{(-\cancel{2})(R+z^2)}{\sqrt{u}} - (\cancel{2})\sqrt{u}\right]_{z^2}^{R^2+z^2}$$

$$= \hat{z}\,\frac{15Qz}{4\pi\varepsilon_0 R^5}\left(-\frac{R^2+z^2}{\sqrt{R^2+z^2}} - \sqrt{R^2+z^2} - \left(\frac{R^2+z^2}{|z|} - |z|\right)\right)$$

- 3c) (10 points) Show that in the limit $z \gg R$, the expression for the electric field reduces to a familiar equation and discuss.

$$= \hat{z}\,\frac{15Qz}{4\pi\varepsilon_0 R^5}\left[\frac{-R^2+z^2}{\sqrt{R^2+z^2}} - \sqrt{R^2+z^2} + \frac{R^2+z^2}{|z|} + |z|\right]$$

if $z \gg R$ $\sqrt{z^2} = |z|$ these 2 terms cancel

$$= \hat{z}\,\frac{15}{4}\frac{Qz}{\pi\varepsilon_0 R^5}\left(R^2+z^2\left(\frac{-1}{\sqrt{R^2+z^2}} + \frac{1}{\sqrt{z^2}}\right)\right)$$

$$= \hat{z}\,\frac{15}{4}\frac{Qz}{\pi\varepsilon_0 R^5}\left(\sqrt{R^2+z^2} + \frac{1}{\sqrt{z^2}}\right) \qquad z \gg R$$

$$= \hat{z}\,\frac{15}{4}\frac{Qz}{\pi\varepsilon_0 R^5}\left(\overset{|z|}{\sqrt{z^2}} + \frac{1}{|z|}\right)$$

$$= \hat{z}\,\frac{15}{4}\frac{Qz}{\pi\varepsilon_0 R^5}\left(1 - \frac{1}{\sqrt{1+\frac{R^2}{z^2}}}\right) = \hat{z}\,\frac{15}{4}\frac{Qz}{\pi\varepsilon_0 R^5}$$

The field should look lie the field due to a pt. charge $q_{in}(r)$