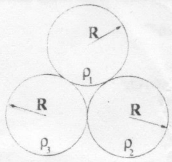
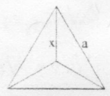
$q_i = \frac{4}{3}\pi R^3 \rho_i$

$r_{ij} = 2R$

$a = 2R$
$x = \frac{2}{\sqrt{3}} R$





$a^2 = 3x^2$

1) Three spheres of identical radius R (but different, uniform charge densities ρ_1, ρ_2 and ρ_3) are arranged so that they touch one another as shown in the diagram above. [The diagram at the right is a hint for later].

- 1a) (10 points) How much work will it take to assemble these spheres into the arrangement shown? [Assume the spheres themselves have already been assembled - that is, neglect the self-energy of each sphere].

$$W_{ext} = \frac{1}{4\pi\epsilon_0} \sum \frac{q_i q_j}{r_{ij}}$$

$$W_{ext} = \frac{1}{4\pi\epsilon_0} \left(\frac{4}{3}\pi R^3\right)^2 \left(\frac{1}{2R}\right) \left[\rho_1\rho_2 + \rho_1\rho_3 + \rho_2\rho_3\right]$$

$$\boxed{W_{ext} = \frac{2\pi R^5}{9\epsilon_0}\left(\rho_1\rho_2 + \rho_1\rho_3 + \rho_2\rho_3\right)}$$

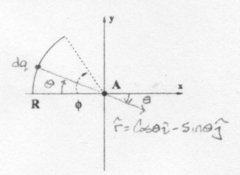
- 1b) (10 points) What is the electric potential at the center of the arrangement?

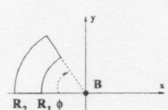
$$V_i = \frac{1}{4\pi\epsilon_0} \frac{q_i}{x}$$

$$V_i = \frac{1}{4\pi\epsilon_0} \frac{4}{3}\pi R^3 \rho_i \frac{\sqrt{3}}{2R}$$

$$V_i = \frac{R^2}{2\sqrt{3}\,\epsilon_0} \rho_i$$

$$V = \sum V_i \Rightarrow \boxed{V = \frac{R^2}{2\sqrt{3}\,\epsilon_0}(\rho_1 + \rho_2 + \rho_3)}$$





$dq = \lambda ds$

$dq = \frac{q}{\phi} d\theta$

- 2a) (10 points) A thin nonconducting rod that carries an electric charge q (uniformly distributed) is bent to form a circular arc of radius R that subtends an angle ϕ as shown in the diagram on the left. Find the electric field (vector) at point A (located at the center of curvature of the arc).

$$d\vec{E} = \frac{dq}{4\pi\epsilon_0 r^2}\hat{r}$$

$$d\vec{E} = \frac{1}{4\pi\epsilon_0 R^2} \frac{q\,d\theta}{\phi} (\cos\theta\,\hat{\imath} - \sin\theta\,\hat{\jmath})$$

$$\vec{E} = \frac{q}{4\pi\epsilon_0 \phi R^2} \int_0^\phi d\theta (\cos\theta\,\hat{\imath} - \sin\theta\,\hat{\jmath})$$

$$\boxed{\vec{E} = \frac{q}{4\pi\epsilon_0 \phi R^2}\left[\sin\phi\,\hat{\imath} - (1-\cos\phi)\,\hat{\jmath}\right]}$$

Alternately,
$\sin\phi = 2\sin\frac{\phi}{2}\cos\frac{\phi}{2}$
$1 - \cos\phi = 2\sin^2\frac{\phi}{2}$

$$\boxed{\vec{E} = \frac{Q\sin\phi/2}{2\pi\epsilon_0 \phi R^2}\left[\cos\frac{\phi}{2}\,\hat{\imath} - \sin\frac{\phi}{2}\,\hat{\jmath}\right]}$$

- 2b) (10 points) Now consider the diagram shown on the right. Charge is spread over a wedge defined by the angle ϕ between the radial distances R_1 and R_2 (as shown) with an area charge density

$$\sigma(r) = \frac{4Q}{\phi(R_2^4 - R_1^4)} r^2$$

where r is the radial distance from the B (located at the center of curvature of the defining arcs). Find the electric field (vector) at B.

$$d\vec{E} = \frac{dq}{4\pi\epsilon_0 \phi r^2}\left[\sin\phi\,\hat{\imath} - (1-\cos\phi)\,\hat{\jmath}\right]$$

$$dq = \sigma dA = \frac{4Q}{\phi(R_2^4 - R_1^4)} r^2 \cdot \phi r dr$$

$$dq = \frac{4Q r^3 dr}{(R_2^4 - R_1^4)}$$

$$\int d\vec{E} = \frac{Q\left[\sin\phi\,\hat{\imath} - (1-\cos\phi)\,\hat{\jmath}\right]}{\pi\epsilon_0 \phi (R_2^4 - R_1^4)} \int_{R_1}^{R_2} dr\, r$$

$$\boxed{\begin{gathered}\vec{E} = \frac{Q}{2\pi\epsilon_0 \phi (R_1^2 + R_2^2)}\left[\sin\phi\,\hat{\imath} - (1-\cos\phi)\,\hat{\jmath}\right] \\ \vec{E} = \frac{Q\sin\phi/2}{\pi\epsilon_0 \phi (R_1^2 + R_2^2)}\left[\cos\frac{\phi}{2}\,\hat{\imath} - \sin\frac{\phi}{2}\,\hat{\jmath}\right]\end{gathered}}$$

Either form is fine

- 1c) (10 points) What is the electric field at the the center of the arrangement?

$$\vec{E}_i = \frac{q_i}{4\pi\epsilon_0 x^2}\hat{r}_i$$

$$\vec{E}_i = \frac{1}{4\pi\epsilon_0} \frac{4}{3}\pi R^3 \rho_i \frac{3}{4R^2}\hat{r}_i$$

$$\vec{E}_i = \frac{R}{4\epsilon_0}\rho_i \hat{r}_i$$

$$\vec{E}_1 = \frac{R}{4\epsilon_0}\left[\qquad -\rho_1\hat{\jmath}\right]$$

$$\vec{E}_2 = \frac{R}{4\epsilon_0}\left[-\rho_2\cos\theta\,\hat{\imath} + \rho_2\sin\theta\,\hat{\jmath}\right]$$

$$\vec{E}_3 = \frac{R}{4\epsilon_0}\left[\rho_3\cos\theta\,\hat{\imath} + \rho_3\sin\theta\,\hat{\jmath}\right]$$

$$\vec{E} = \sum \vec{E}_i$$

$$\vec{E} = \frac{R}{4\epsilon_0}\left[(\rho_3 - \rho_2)\cos\theta\,\hat{\imath} + \left[(\rho_3 + \rho_2)\sin\theta - \rho_1\right]\hat{\jmath}\right]$$

$\theta = 30° \Rightarrow \cos(30°) = \sqrt{3}/2 \qquad \sin(30°) = 1/2$

$$\boxed{\vec{E} = \frac{R}{8\epsilon_0}\left[\sqrt{3}(\rho_3 - \rho_2)\,\hat{\imath} + (\rho_3 + \rho_2 - 2\rho_1)\,\hat{\jmath}\right]}$$

- 2b) *(continued...)*

- 2c) (10 points) Find the electric potential produced by the wedge at point B relative to a point infinitely-distant from the wedge.

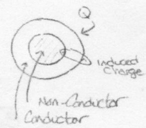
All the points in a thin arc are equidistant to B, So this is easier than it looks :)

$$dV = \frac{dq}{4\pi\epsilon_0 r} = \frac{1}{4\pi\epsilon_0 r} \cdot \frac{4Q r^3 dr}{(R_2^4 - R_1^4)}$$

$$\int dV = \frac{Q}{\pi\epsilon_0 (R_2^4 - R_1^4)} \int_{R_1}^{R_2} dr\, r^2$$

$$\boxed{V = \frac{Q\,(R_2^3 - R_1^3)}{3\pi\epsilon_0 (R_2^4 - R_1^4)}}$$

3) A spherical charge distribution of radius R carries a volume charge density



$$\rho(r) = \rho_0\,(1 - \frac{r^3}{R^3})$$

It is surrounded by a concentric spherical conducting shell that extends from $r = R$ to $r = 2R$ and carries an excess charge Q.

- 3a) (10 points) Find the charge inside a concentric sphere of radius r, for all values of r. Also, find the surface charge densities on the inner and outer surfaces of the conducting shell.

Spherical dist: $dq = \rho dv = \rho\, 4\pi r^2 dr$

$(r<R)$ $dq = \rho dV = \rho_0 (1 - \frac{r^3}{R^3}) 4\pi r^2$

$q_{in}(r) = \int dq = 4\pi\rho_0 \int_0^r (r^2 - r^5/R^3)\,dr$

$q_{in}(r) = 4\pi\rho_0 \frac{1}{6}\left[2r^3 - \frac{r^6}{R^3}\right]$

$q_{in}(r) = \frac{2}{3}\pi\rho_0 r^3 (2 - r^3/R^3)$

$(R<r<2R)$ $q_{in}(r) = 0$

$(2R<r)$ $q_{in}(r) = q_{in}(R) + Q$

$q_{in}(r) = \frac{2}{3}\pi\rho_0 R^3 + Q$

$\sigma_{in}(R) = \frac{-q_{in}(R)}{4\pi R^2} = -\frac{\rho_0 R}{6}$

$\sigma_{in}(2R) = \frac{q_{in}(R) + Q}{4\pi (2R)^2} = \frac{\frac{2\pi\rho_0 R^3}{3} + Q}{16\pi R^2}$

$$\boxed{\begin{aligned} q_{in}(r) &= \tfrac{2\pi}{3}\rho_0 r^3 (2 - r^3/R^3) && (r<R)\\ q_{in}(r) &= 0 && (R<r<2R)\\ q_{in}(r) &= \tfrac{2\pi}{3}\rho_0 R^3 + Q && (2R<r)\\ \sigma_{in}(R) &= -\tfrac{\rho_0 R}{6} \\ \sigma_{in}(2R) &= \tfrac{\rho_0 R}{24} + \tfrac{Q}{16\pi R^2} \end{aligned}}$$

- 3b) (10 points) Find the electric field as a function of the radial distance from the center of the charge distribution (r) for all values of r.

<u>Spherical Symmetry:</u> $\vec{E}(r) = \frac{q_{in}(r)}{4\pi\epsilon_0 r^2}\hat{r}$

$$\boxed{\begin{aligned} &(r<R) && \vec{E} = \tfrac{\rho_0}{6\epsilon_0} r (2 - r^3/R^3)\hat{r}\\ &(R<r<2R) && \vec{E} = 0\\ &(2R<r) && \vec{E} = \tfrac{1}{4\pi\epsilon_0 r^2}\left(\tfrac{2\pi\rho_0 R^3}{3} + Q\right)\hat{r} \end{aligned}}$$

- 3c) (10 points) If the electric potential within the conductor is given as V_0, find the potential as a function of the radial distance from the center of the charge distribution (r) for all values of r.

Spherical Symmetry – $\vec{E}\cdot d\vec{r} = E_r dr$

$\Delta V(r_{ref}, r) = -\int_{r_{ref}}^r \vec{E}\cdot d\vec{r}$

$V(r) = V(r_{ref}) - \int_{r_{ref}}^r E_r dr$

$V(r) = V_0 - \int_{r_{ref}}^r E_r dr$

$(r<R)$ $V(r) = V_0 - \int_R^r \frac{\rho_0}{6\epsilon_0} r (2 - r^3/R^3)\,dr$

$V(r) = V_0 - \frac{\rho_0}{6\epsilon_0}\left[r^2 - \frac{1}{5} r^5/R^3\right]_R^r$

$V(r) = V_0 - \frac{\rho_0 R^2}{30\epsilon_0}\left[5 r^2/R^2 - r^5/R^5 - 4\right]$

$(R<r<2R)$ $V(r) = V_0 - \int_R^r 0\,dr$

$V(r) = V_0$

$(2R<r)$ $V(r) = V_0 - (\frac{2\pi\rho_0 R^3}{3} + Q)\frac{1}{4\pi\epsilon_0}\int_R^r \frac{dr}{r^2}$

$V(r) = V_0 + \frac{1}{4\pi\epsilon_0}\left(\frac{2\pi\rho_0 R^3}{3} + Q\right)\left(\frac{1}{r} - \frac{1}{R}\right)$

$$\boxed{\begin{aligned} V(r) &= V_0 + \tfrac{\rho_0 R^2}{30\epsilon_0}\left[4 - 5 r^2/R^2 + r^5/R^5\right] && (r<R)\\ V(r) &= V_0 && (R<r<2R)\\ V(r) &= V_0 + \tfrac{1}{4\pi\epsilon_0}\left[\tfrac{2\pi\rho_0 R^3}{3} + Q\right]\left(\tfrac{1}{r} - \tfrac{1}{R}\right) && (2R<r) \end{aligned}}$$