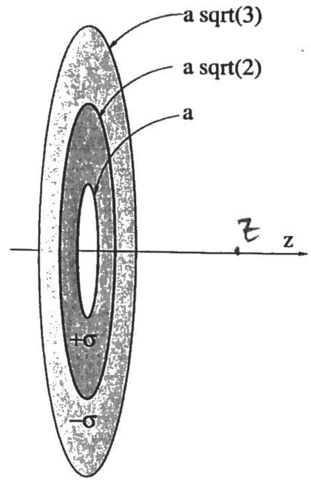



1) The washer shown above lies in the $x, y$-plane, centered on the origin. From $r = a$ to $r = \sqrt{2}\,a$, it carries a uniform surface charge density $\sigma$. From $r = \sqrt{2}\,a$ to $r = \sqrt{3}\,a$, it carries a uniform surface charge density $-\sigma$.

- 1a) (15 points) Find the electric potential (with respect to infinity) at all points on the positive $z$-axis.

$$dV_{ring} = \frac{k\,dq}{r} = \frac{k\,dq}{\sqrt{R^2+z^2}}$$

$$V_{ring} = \int \frac{k\,dq}{\sqrt{R^2+z^2}} = \frac{kQ}{\sqrt{R^2+z^2}}$$

$$dV_{disk\,(\sigma)} = \frac{k\,dq}{\sqrt{r^2+z^2}} = \frac{k\sigma(2\pi r)\,dr}{\sqrt{r^2+z^2}}$$

$$V_{disk\,(\sigma)} = \int_a^{\sqrt{2}a} \frac{k\sigma(2\pi r)\,dr}{\sqrt{r^2+z^2}} = k\sigma\pi \int_a^{\sqrt{2}a} \frac{2r\,dr}{\sqrt{r^2+z^2}} = k\sigma\pi \int_{a^2+z^2}^{(\sqrt{2}a)^2+z^2} \frac{du}{u^{1/2}}$$

$$u = r^2+z^2 \qquad du = 2r\,dr$$

$$\frac{1}{2\cancel{4}\cancel{\pi}\varepsilon_0}\cdot\sigma\cancel{\pi}\cdot\cancel{2} = \frac{\sigma}{2\varepsilon_0}$$

$$= k\sigma\pi\left(2u^{1/2}\right)\Big|_{a^2+z^2}^{2a^2+z^2} = \frac{\sigma}{2\varepsilon_0}\left(\sqrt{2a^2+z^2} - \sqrt{a^2+z^2}\right)$$

$$V_{disk\,(-\sigma)} = \int_{\sqrt{2}a}^{\sqrt{3}a} \frac{k(-\sigma)\,2\pi r\,dr}{\sqrt{r^2+z^2}} = -k\sigma\pi\left(2u^{1/2}\right)\Big|_{2a^2+z^2}^{3a^2+z^2}$$

$$= -\frac{\sigma}{2\varepsilon_0}\left(\sqrt{3a^2+z^2} - \sqrt{2a^2+z^2}\right)$$

15

$$V_{washer} = \frac{\sigma}{2\varepsilon_0}\left(\sqrt{2a^2+z^2} - \sqrt{a^2+z^2} + \sqrt{2a^2+z^2} - \sqrt{3a^2+z^2}\right)$$

$$\boxed{V_{washer} = \frac{\sigma}{2\varepsilon_0}\left(2\sqrt{2a^2+z^2} - \sqrt{a^2+z^2} - \sqrt{3a^2+z^2}\right)}$$

- 1b) (10 points) Find the (vector) electric field at all points on the positive $z$-axis (be very clear how you obtain each component!)

$E_x$ and $E_y = 0$ by symmetry

$$\Delta V = -\int \vec{E} \cdot d\vec{s}$$

$$\vec{E} = -\vec{\nabla} V$$

$$E_z = -\frac{\partial V}{\partial z} = -\frac{\sigma}{2\varepsilon_0}\left[\frac{2z}{(2a^2+z^2)^{1/2}} - \frac{\cancel{2}z}{\cancel{2}(a^2+z^2)^{1/2}} - \frac{\cancel{2}z}{\cancel{2}(3a^2+z^2)^{1/2}}\right]$$

$$\boxed{\vec{E} = -\frac{\sigma}{2\varepsilon_0}\left[\frac{2z}{(2a^2+z^2)^{1/2}} - \frac{z}{(a^2+z^2)^{1/2}} - \frac{z}{(3a^2+z^2)^{1/2}}\right]\hat{k}}$$

10

- 1c) (5 points) Find the electric potential (with respect to infinity) at points along the positive $z$ axis that are very distant from the washer [**Hint:** $(1+x)^n \approx 1+nx+\frac{1}{2}n(n-1)x^2$]. Discuss the monopole, dipole and quadrapole moments for this charge distribution.

$z >> a$

$\frac{z}{a} >> 1$, we want $\frac{a}{z}$

$$V \approx \frac{\sigma}{2\varepsilon_0}\left[2\sqrt{2}a\left(1+\left(\frac{z}{\sqrt{2}a}\right)^2\right)^{1/2} - a\left(1+\left(\frac{z}{a}\right)^2\right)^{1/2} - \sqrt{3}a\left(1+\left(\frac{z}{\sqrt{3}a}\right)^2\right)^{1/2}\right]$$

$$\approx \frac{\sigma}{2\varepsilon_0}\left[2\sqrt{2}a\left(1+\frac{z^2}{4a^2}\right) - a\left(1+\frac{z^2}{2a^2}\right) - \sqrt{3}a\left(1+\frac{z^2}{6a^2}\right)\right]$$

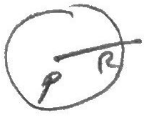
$$\approx \frac{\sigma}{2\varepsilon_0}\left[(2\sqrt{2}-1-\sqrt{3})a\right.$$

keep higher order terms

2

Consider a very long uniform cylindrical distribution of charge, of volume charge-density $\rho$ and radius $R$...

- 2a) (10 points) Find the electric field (vector) for all points inside and outside the cylinder.




$$\boxed{\vec{E} = \begin{cases} \frac{\rho r}{2\varepsilon_0}\hat{r}, & 0 \le r \le R \\ \frac{\rho R^2}{2\varepsilon_0 r}\hat{r}, & r > R \end{cases}}$$

$0 \le r \le R$:

$$\frac{q_{enc}}{\varepsilon_0} = \oint \vec{E}\cdot d\vec{A}$$

$$\frac{\rho(\pi r^2 \cdot L)}{\varepsilon_0} = E(2\pi r \cdot L)$$

$$E = \frac{\rho r}{2\varepsilon_0}$$

$r > R$:

$$\frac{q_{enc}}{\varepsilon_0} = \oint \vec{E}\cdot d\vec{A}$$

$$\frac{\rho(\pi R^2 L)}{\varepsilon_0} = E(2\pi r L)$$

$$E = \frac{\rho R^2}{2\varepsilon_0 r}$$

$\frac{10}{10}$

- 2b) (10 points) Find the electric potential for all points inside and outside the cylinder, given the potential at the center of the sphere is $V_0$.

$0 \le r \le R$:

$$V_0 + \Delta V = V_0 - \int_0^r \frac{\rho r}{2\varepsilon_0}\,dr$$

$$= V_0 - \frac{\rho}{2\varepsilon_0}\left(\frac{r^2}{2}\right)\Big|_0^r$$

$$= V_0 - \frac{\rho r^2}{4\varepsilon_0}$$

$$\boxed{V = \begin{cases} V_0 - \frac{\rho r^2}{4\varepsilon_0}, & 0 \le r \le R \\ V_0 - \frac{\rho R^2}{4\varepsilon_0} + \frac{\rho R^2}{2\varepsilon_0}\ln\left(\frac{R}{r}\right), & r > R \end{cases}}$$

$r > R$:

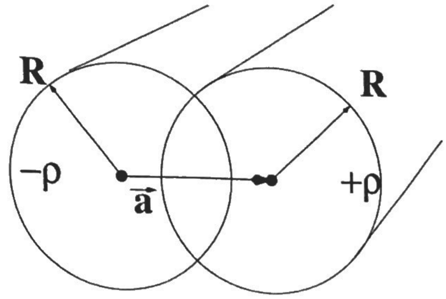
$$V_0 - \frac{\rho R^2}{4\varepsilon_0} + \Delta V$$

$$\Delta V = -\int_R^r \frac{\rho R^2}{2\varepsilon_0 r}$$

$$= -\frac{\rho R^2}{2\varepsilon_0}(\ln r)\Big|_R^r$$

$$= \frac{\rho R^2}{2\varepsilon_0}\ln\left(\frac{R}{r}\right)$$

$\frac{10}{10}$




- 2c) (10 points) Suppose a pair of cylinders, each of radius $R$, are placed so that their symmetry axes are parallel and are separated by a vector displacement $\vec{a}$ as shown. If one cylinder has a uniform volume charge density $+\rho$ and the other cylinder has a uniform volume charge density $-\rho$, find the electric field (vector) in the region where the cylinders overlap and show that it is uniform.

overlapping region:

$$\frac{q_{enc}}{\varepsilon_0} = \oint \vec{E} \cdot d\vec{A}$$

$\frac{5}{10}$

at different pts the volumes aren't the same

$$\frac{\rho V_{overlap} - \rho V_{overlap}}{\cancel{\varepsilon_0}} = E \oint dA$$

$$0 = E \oint dA$$

$$\boxed{E = 0 \quad (\text{uniform})}$$

IV

Consider a diffuse, spherically-symmetric charge distribution of radius $R$ and charge $Q$ whose volume charge density varies as $r^4$ ($r$ being the distance from the center of the distribution).

- 3a) (15 pts) Obtain functions that describe the volume charge density of the distribution and $q_{in}(r)$ (the total charge contained in a sphere of radius $r$, placed concentric with the charge distribution), in terms of $Q$, $R$ and $r$, for all $r$.

$$\rho(r) = C r^4$$

$$Q = \int_0^R C r^4 (4\pi r^2) dr$$

$$Q = 4\pi C \int_0^R r^6 dr$$

$$Q = 4\pi C \left(\frac{r^7}{7}\right)\Big|_0^R$$

$$Q = 4\pi C \left(\frac{R^7}{7}\right)$$

$$C = \frac{7Q}{4\pi R^7}$$

$$\boxed{\rho(r) = \begin{cases} \frac{7Q}{4\pi R^7} r^4, & 0 \le r \le R \\ 0, & r > R \end{cases}}$$

✓

$$q_{in}(r) = \int_0^r C r^4 (4\pi r^2) dr$$

$$= C(4\pi) \int_0^r r^6 dr$$

$$= 4\pi C \frac{r^7}{7}$$

$$\boxed{q_{in}(r) = \begin{cases} \frac{Q r^7}{R^7}, & 0 \le r \le R \\ \cancel{X}Q, & r > R \end{cases}}$$

✓

$$\frac{\cancel{4\pi}}{\cancel{7}} \cdot \frac{\cancel{7}Q}{\cancel{4\pi} R^7}$$

+12

- 3b) (15 pts) How much work would one have to do to assemble this distribution from a reservoir of charge sitting some infinite distance away?

$dW_{ext} = dV dq\ V(r)$

$dq = \rho(r)(4\pi r^2)\,dr \qquad V(r) = \frac{k\, q_{in}(r)}{r}$

$dq = \frac{7Q}{\cancel{4\pi} R^7} r^4 (\cancel{4\pi} r^2)\,dr \qquad = \frac{k\, Q r^7}{r R^7} = \frac{kQ}{R^7} r^6$

$dq = \frac{7Q}{R^7} r^6\,dr$

$$W_{ext} = \int dW_{ext} = \int_0^R \frac{7kQ^2}{R^{14}} r^{12}\,dr$$

$$= \frac{7kQ^2}{R^{14}} \int_0^R r^{12}\,dr$$

$$= \frac{7kQ^2}{\cancel{R^{14}}}\left(\frac{\cancel{R^{13}}}{13}\right)$$

$$\begin{array}{r} 1 \\ 13 \\ \times 4 \\ \hline 52 \end{array}$$

$$= \frac{7Q^2}{4 \cdot 13\pi\varepsilon_0} = \boxed{\frac{7Q^2}{52\pi\varepsilon_0}} = W_{ext}$$

✓