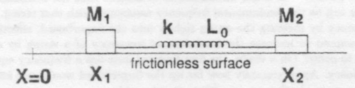


1) A pair of masses, M_1 and M_2 are joined by a spring of constant k and natural (unstretched) length L_0.

- 1a) (5 points) Find the amount the spring is extended or compressed (Δx) as a function of the position of each mass (x_1 and x_2, respectively). Make sure Δx is positive when the spring is extended and negative when it's compressed.

$$\boxed{\Delta x = x_2 - x_1 - L_0}$$

- 1b) (5 points) Use Newton's laws to obtain differential equations (written in terms of x_1 and x_2) that describe the motion of each mass. Note that these are 'coupled' equations - they will each depend on both x_1 and x_2 - but not to worry, we'll address that later. Make sure the derivative term in each equation has the correct sign when the spring is extended and compressed!

$\sum F_x = Ma_x$:

$F_{sp,1} = k\Delta x$ ← positive when spring is stretched

$F_{sp,2} = -k\Delta x$ ← negative when spring is stretched

$$k(x_2 - x_1 - L_0) = m_1 \frac{d^2x_1}{dt^2}$$

$$-k(x_2 - x_1 - L_0) = m_2 \frac{d^2x_2}{dt^2}$$

in Standard form:

$$\boxed{\frac{d^2x_1}{dt^2} + \frac{k}{m_1}(x_1 - x_2) = -\frac{kL_0}{m_1} \qquad \frac{d^2x_2}{dt^2} + \frac{k}{m_2}(x_2 - x_1) = \frac{kL_0}{m_2}}$$

- 1c) (5 points) Algebraically relate the derivative term in x_1 to the derivative term in x_2. Use this result, along with your answer to part a, to obtain relationships between a derivative of Δx and each of the derivatives of x_1 and x_2 that appear in your answers to part b.

from 1b) $-m_1 \frac{d^2x_1}{dt^2} = m_2 \frac{d^2x_2}{dt^2}$ from 1a) $\frac{d^2\Delta x}{dt^2} = \frac{d^2x_2}{dt^2} - \frac{d^2x_1}{dt^2}$

$$\boxed{\frac{d^2x_1}{dt^2} = \frac{-M_2}{M_1 + M_2}\frac{d^2\Delta x}{dt^2} \qquad \frac{d^2x_2}{dt^2} = \frac{M_1}{M_1 + M_2}\frac{d^2\Delta x}{dt^2}}$$

- 1d) (10 points) Now it's time to put it all together - rewrite the differential equations from part b in terms of Δx. The results should look familiar. Find the solution for Δx as a function of time (make sure you evaluate, in terms of given information, any constants that are determined by the construction of the system).

$$k(x_2 - x_1 - L_0) = m_1 \frac{d^2x_1}{dt^2}$$

$$k\Delta x = -\frac{m_1 m_2}{m_1 + m_2}\frac{d^2\Delta x}{dt^2}$$

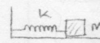
$$-k(x_2 - x_1 - L_0) = m_2 \frac{d^2x_2}{dt^2}$$

$$-k\Delta x = \frac{m_1 m_2}{m_1 + m_2}\frac{d^2\Delta x}{dt^2}$$

$\Rightarrow$

$$\boxed{\frac{d^2\Delta x}{dt^2} + \frac{k(m_1 + m_2)}{m_1 m_2}\Delta x = 0 \qquad \Delta x = A\cos(\omega t + \phi) \qquad \omega = \sqrt{k/\mu} \qquad \mu = \frac{m_1 m_2}{m_1 + m_2}}$$

- 1e) (5 points) Suppose we were to tie one end of the spring off to a wall, and the other end of the spring to a mass M. What value would M have to have in order for this new system to oscillate with the same period as the two-mass system we've been working on? This value, known as the *reduced mass*, is used to simplify the discussion of binary systems (diatomic molecules, for instance) in oscillation.



$\omega = \sqrt{\frac{k}{m}} \Rightarrow$

$$\boxed{M = \mu = \frac{M_1 M_2}{M_1 + M_2}}$$

2) Recall that the amplitude of a driven mass-spring system is given by

$$A(\Omega) = \frac{F_0/m}{\sqrt{(\omega_0^2 - \Omega^2)^2 + (\frac{b\Omega}{m})^2}}$$

- 2a) (10 points) Show that the following relationship holds true for a mass-spring system:

$$\omega_{damp}^2 = \frac{1}{2}\left(\Omega_{res}^2 + \omega_0^2\right)$$

$$\Omega_{res} = \sqrt{\omega_0^2 - 2\left(\frac{b}{2m}\right)^2}$$

$$\omega_{damp} = \sqrt{\omega_0^2 - \left(\frac{b}{2m}\right)^2}$$

$$\Omega_{res}^2 - 2\omega_{damp}^2 = -\omega_0^2$$

$$\boxed{\omega_{damp}^2 = \frac{1}{2}\left(\Omega_{res}^2 + \omega_0^2\right)}$$

$$T = \frac{2\pi}{\omega} = \frac{1}{f}$$

$$\omega = 2\pi f$$

$$\omega_{damp} = \omega_1 = 2\pi f_1$$

- 2b) (5 points) Convince the grader that the relationship in part a holds for all simple harmonic oscillators.

In general, simple harmonic oscillators satisfy

$$\frac{d^2x}{dt^2} + 2\beta\frac{dx}{dt} + \omega_0^2 x = F(t)$$

For the mass-spring system, $2\beta = \frac{b}{m} \Rightarrow \beta = \frac{b}{2m}$

For any other system, you can think of $\frac{b}{2m}$ as a placeholder for the relevant β...

- 2c) (10 points) On the way to class, you spot an abandoned bird's nest sitting in a low-hanging, more-or-less horizontal branch. You displace the end by some small amount and release it, and note that a) the tip makes about f_1 complete cycles every second, and b) it takes about N complete cycles for the amplitude of the branch's vibrations to drop to half its initial value. Find the natural frequency (f_0) for the branch/nest system. [*Careful.* These are f's, not ω's. f's are easier to observe directly.]

We'll write $b/2m$ for β, as that is probably more familiar...

$$A = A_0 e^{-\frac{bt}{2m}}$$

$$\frac{1}{2} = e^{-\frac{bNT}{2m}}$$

$$\ln(2) = N\frac{b}{2m}\frac{1}{f_1}$$

$$\frac{b}{2m} = \frac{f_1 \ln(2)}{N}$$

$$\omega_1^2 = \omega_0^2 - \left(\frac{b}{2m}\right)^2$$

$$f_1^2 = f_0^2 - \left(\frac{1}{2\pi}\right)^2\left(\frac{f_1 \ln(2)}{N}\right)^2$$

$$f_0^2 = f_1^2\left[1 + \left(\frac{\ln(2)}{2\pi N}\right)^2\right]$$

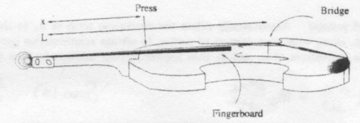
$$\boxed{f_0 = f_1\sqrt{1 + \left(\frac{\ln(2)}{2\pi N}\right)^2}}$$

- 2d) (5 points) Suppose we want to knock that old nest out of the tree. What would be the most effective frequency to shake the branch at?

$$\Omega_{res}^2 = 2\omega_{damp}^2 - \omega_0^2$$

$$f_{res}^2 = 2f_1^2 - f_1^2\left(1 + \left(\frac{\ln(2)}{2\pi N}\right)^2\right)$$

$$\boxed{f_{res} = f_1\sqrt{1 - \left(\frac{\ln(2)}{2\pi N}\right)^2}}$$



- 3a) (5 points) (Carefully) pluck a string on a violin. For a brief moment, the pluck generates noise - but that noise quickly gives way to music. Explain, in terms of physics, what is happening.

That brief, arbitrary, pluck divides its energy over a broad swath of harmonics. For an instant, that superposition of many harmonics is not unlike the sound of a cheap (and harmonic-rich) musical birthday card. But most of the higher harmonics don't get much energy and they damp out quickly. We are left with a few strong lower harmonics, and the tone we expect from music.

- 3b) (5 points) *Derive* the set of resonance frequencies for one of the open (that is, un-fingered) strings in terms of its effective length (L), the volume mass density of the material it is made of (ρ), the diameter of the string (D) and the tension in the string (T).

$N\frac{\lambda}{2} = L$ ⤸ $f\lambda = v_x$

$f_N = N\frac{v_x}{2L}$ ⤸ $v_x = \sqrt{\frac{T}{\mu}}$

$f_N = N\frac{1}{2L}\sqrt{\frac{T}{\mu}}$ ⤸ $\mu = M/L$

$f_N = \frac{N}{2}\sqrt{\frac{T}{mL}}$ ⤸ $m = \frac{1}{4}\pi\rho D^2 L$

$f_N = \frac{N}{2}\sqrt{\frac{4T}{\pi\rho D^2 L^2}}$

$$f_N = N\frac{1}{DL}\sqrt{\frac{T}{\pi\rho}}$$

- 3c) (5 points) It is not unreasonable to assume that the dominant frequency heard from an excited string will be the fundamental frequency associated with that string. One may change the fundamental frequency by pressing the string tightly into the fingerboard, effectively changing its length. Suppose you wanted to increase the fundamental frequency of a string by a factor F. Where (x) would you have to press? On a violin, the first fingered note has a frequency equal to 1.12 times the open-stringed frequency. Approximately how far up the fingerboard would you have to press to generate it?

(x is defined in the picture)

$f \propto \frac{1}{L}$

$\frac{f_{new}}{f} = \frac{L}{L_{new}}$

$\frac{Ff}{f} = \frac{L}{L-x}$

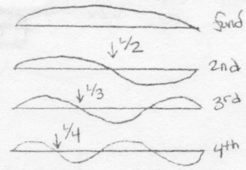
$x = L\left(\frac{F-1}{F}\right)$

To raise the pitch of an open string by a factor F, one must press a distance $x = L\left(\frac{F-1}{F}\right)$ from the top of the fingerboard.

If $F = 1.12$, $x = \frac{0.12}{1.12}L \simeq 11\% L$

⇒ you'd have to press about 11% of the way down from the top of the fingerboard

- 3d) (10 points) Another way to change the dominant frequency you hear is to **lightly** press the string in some magic spot. Since the string is not tightly pressed, it is still able to vibrate on either side of the finger - this has the effect of imposing an intermediate node on the system, emphasizing some harmonic over the fundamental. Find each of these magic spots (x) and the frequency associated with it (in terms of the fundamental).



Lightly pressing $\frac{L}{N}$ from the top of the fingerboard emphasizes the N^{th} harmonic

- 3e) (5 points) Bowing near the bridge can make a much nicer sound than bowing off towards the fingerboard. Explain in terms of physics why this is so.

Bowing near the bridge imposes less in the way of intermediate 'boundary' conditions - bowing away from the bridge introduces a wide, sloppy antinode some place where it might not want to be, resulting in lots of driven harmonics (that don't damp out so quickly!)