Final Exam 1BW22

Full Name (Printed) _ ____________________

Full Name (Signature) _ ____________________

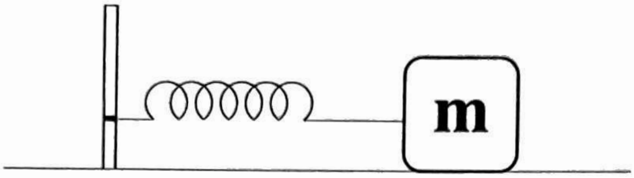
Student ID Number _ ____________________

- The exam is open-book and open notes. You will probably do better to limit yourself to a single page of notes you prepared well in advance.
- **All work must be your own.** You are not allowed to collaborate with anyone else, you are not allowed to discuss the exam with anyone until all the exams have been submitted (after the close of the submissions window for the exam).
- You have **120 minutes** to complete the exam and more than sufficient time to scan the exam and upload it to GradeScope. The exam *must* be uploaded to GradeScope within the time alloted (that is, by the end of the 3-hour finals slot). We will only except submissions through GradeScope and will not accept any exam submitted after the submission window closes (CAE students must contact Corbin for instructions).
- **Given the limits of GradeScope, you must fit your work for each part into the space provided.** You may work on scratch paper, but you will not be able to upload the work you do on scratch paper, so it is essential that you copy your complete solution onto the exam form for final submission. We can only consider the work you submit on your exam form.
- **For full credit the grader must be able to follow your solution from first principles to your final answer.** ***There is a valid penalty for confusing the grader.***
- It is YOUR responsibility to make sure the exam is scanned correctly and uploaded before the end of the submission window. The graders may refuse to grade pages that are significantly blurred, solutions to problems that are not written in the correct place, pages submitted in landscape mode and/or work that is otherwise illegible - if any of this occurs, you may not receive *any* credit for the affected parts.
- Focus on the concepts involved in the problem, the tools to be used, and the set-up. If you get these right, all that's left is algebra.
- **Have Fun!**

The following must be signed before you submit your exam:

By my signature below, I hereby certify that all of the work on this exam was my own, that I did not collaborate with anyone else, nor did I discuss the exam with anyone while I was taking it.

Signature ______________________________



EXAM 1) A block of mass m is attached to a spring (of unknown constant) and placed on a horizontal surface, as shown. Once set in motion, it is observed that the block oscillates with an angular frequency ω and loses 90% of its energy every 50 complete cycles.

- **1a) (10 points)** Find the natural (angular) frequency for the oscillator

$$m\frac{d^2x}{dt^2} = -kx - bv \Rightarrow A_0 e^{-\beta t}\cos(\omega t + \phi) = A(t)$$

$2\pi f = \omega \Rightarrow T = \frac{2\pi}{\omega} \therefore$ loses 90% energy in $\frac{100\pi}{\omega}$

$$\cancel{A_0} e^{-\beta \frac{100\pi}{\omega}} = 0.1\cancel{A_0} \qquad \beta = -\frac{\omega \ln(0.1)}{100\pi}$$

$$\therefore \omega = \sqrt{\omega_0^2 - \beta^2} \qquad \sqrt{\omega^2 + \beta^2} = \omega_0$$

$$\omega_0 = \sqrt{\omega^2 + \left(\frac{\omega \ln(0.1)}{100\pi}\right)^2}$$

- **1b) (10 points)** At what (angular) frequency would you have to drive the oscillator to obtain the largest amplitude vibrations?

$$\Omega_{res} = \sqrt{\omega_0^2 - 2\beta^2} = \sqrt{\omega^2 + \left(\frac{\omega\ln(0.1)}{100\pi}\right)^2 - 2\left(\frac{\omega\ln(0.1)}{100\pi}\right)^2}$$

$$= \sqrt{\omega^2 - \left(\frac{\omega\ln(0.1)}{100\pi}\right)^2}$$

$$A(\Omega) = \frac{F}{\sqrt{(\omega_0^2 - \Omega^2)^2 + (2\beta\Omega)^2}}$$

$$y = (\omega_0^2 - \Omega)^2 + (2\beta\Omega)^2 \qquad \frac{dy}{d\Omega} = 2(\omega_0^2 - \Omega^2)\cdot -2\Omega + 2(2\beta)^2\Omega = 0$$

$$\Rightarrow \Omega_{res} = \sqrt{\omega_0^2 - 2\beta^2}$$

• 1c) (10 points) Suppose the oscillator is driven by a sinusoidal force of amplitude $F_0 = ma$, where a is some constant with the same dimensions as acceleration. What is the largest displacement amplitude that can be obtained with the oscillator?

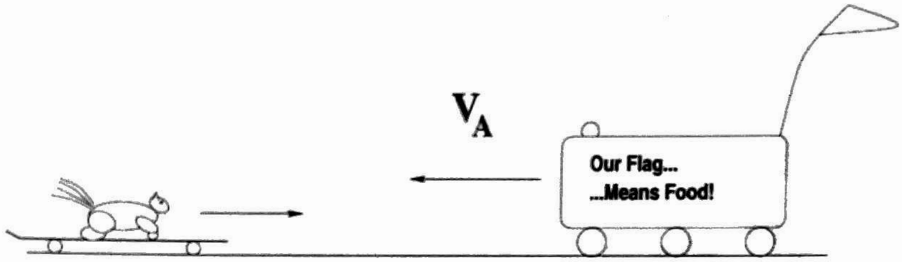
$$x(t) = A_0 e^{-\beta t}\cos(\omega t) + \frac{ma}{\sqrt{(\omega_0^2-\Omega^2) + (2\beta\Omega)^2}}\cos(\Omega t)$$

Set Ω to Ω_{res} and $t=0$ for largest displacement amplitude

then largest displacement is $A_0 + \frac{ma}{\sqrt{(\omega_0^2-\Omega_{res}^2)^2+(2\beta\Omega_{res})^2}}$

$$A_0 + \frac{ma}{\sqrt{\left(\omega_0^2 - \omega^2 + \left(\frac{\omega\ln(0.1)}{100\pi}\right)^2 + \left(\frac{2\omega\ln(0.1)}{100\pi}\right)^2\left(\omega^2 - \left(\frac{\omega\ln(0.1)}{100\pi}\right)\right)\right)^2}}$$

$$= A_0 + \frac{ma}{\sqrt{2\left(\frac{\omega\ln(0.1)}{100\pi}\right)^2 + \left(\frac{2\omega\ln(0.1)}{100\pi}\right)^2\left(\omega^2-\left(\frac{\omega\ln(0.1)}{100\pi}\right)\right)^2}}$$



EXAM 2) A squirrel on a longboard passes an oncoming food-bot approaching on a parallel but directly-adjacent path. The food-bot's speed (relative to the ground) is V_A. As the food-bot approaches, the squirrel hears warning beeps with a frequency f_1. As the food-bot recedes, the squirrel hears the same beeps, but now with a frequency f_2.

- 2a) (10 points) At what frequency are the warning beeps actually being emitted?

$$f_1 = \frac{V_{snd} + V_{squirrl}}{V_{snd} - V_A} f_{src} \qquad f_2 = \frac{V_{snd} + V_{squirrl}}{V_{snd} + V_A} f_{src}$$

$$\frac{f_1}{f_{src}} (V_{snd} - V_A) = V_{snd} + V_{squirrl} \qquad f_{src} = \frac{f_2 (V_{snd} + V_A)}{V_{snd} + V_{squirrl}}$$

$$f_2 = \frac{f_1 (V_{snd} - V_A) - f_{src} V_{snd} + V_{snd} f_{src}}{V_{snd} + V_A}$$

$$f_{src} \left(f_1 (V_{snd} - V_A) \right) - f_{src} \ldots (V_{snd} + V_A)$$

- 2b) (10 points) How fast is the squirrel moving relative to the ground?

$$V_{squirrel} = \frac{f_1}{f_{src}}(V_{snd} - V_A) - V_{snd} = f_1, \quad f_1(V_{snd} -$$

Equation from part 1 is 2 equations w/ 2 unknowns $V_{squirrel}$ and f_{src}.

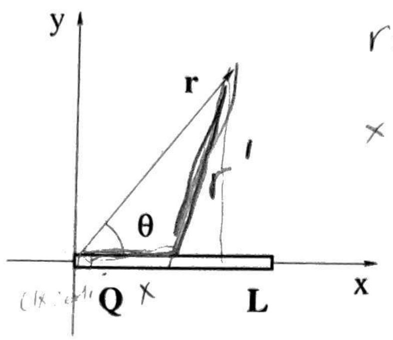
Take answer from part a (couldn't find f_{src} :)) and plug into system of equations to get $V_{squirrel}$

- 2c) (10 points) What would it mean if f_1 had the same value as f_2? Evaluate your previous answers from the previous parts in this limit and check for consistency.

If $f_1 = f_2$, then $V_A = 0$, the bot would not be moving

$$f_1 = f_2 \Rightarrow \frac{V_{snd} + V_{squirrel}}{V_{snd} - V_A} = \frac{V_{snd} + V_{squirrel}}{V_{snd} + V_a}$$

$$\Rightarrow -V_a = V_a \Rightarrow V_a = 0$$



EXAM 3) A thin uniform rod lies on the x-axis with one end on the origin and the other at $x = L$. The rod carries a total charge Q, uniformly distributed. $\lambda = \frac{Q}{L}$

You may take the following as given (for small values of x):

$$(1 + ax + bx^2)^{-\frac{1}{2}} \approx 1 - \frac{1}{2}ax + \frac{1}{2}\left(\frac{3}{4}a^2 - b\right)x^2$$

- 3a) (15 points) Find the electric potential at a point described by r and θ for which $r \gg L$. Consider using the Taylor expansion given above.

$$dV = \frac{dq}{4\pi\epsilon_0 r} = \frac{\lambda\, dx}{4\pi\epsilon_0 r'} \Rightarrow \frac{\lambda\, dx}{4\pi\epsilon_0 r'} \cdot \frac{(x - \cos\theta)}{r'^{1/2}}$$

$r' = r^2 + x^2 - 2xr\cos\theta$ law of cosines

$$dV = \frac{\lambda(x-\cos\theta)\, dx}{4\pi\epsilon_0 (r^2 + x^2 - 2xr\cos\theta)^{3/2}} \quad \int_0^V dV = \int_0^L \frac{\lambda (x\cos\theta)\, dx}{4\pi\epsilon_0 r'^{3/2}}$$

$$= \frac{\lambda}{4\pi\epsilon_0 r^2}\int_0^L \frac{(x-\cos\theta)\, dx}{\left(1 + \frac{1}{r^2}x^2 - \frac{2\cos\theta}{r}x\right)^{3/2}} \qquad du = \left(\frac{2x}{r^2} - \frac{2\cos\theta}{r}\right)dx$$

$$\frac{\lambda}{4\pi\epsilon_0 r^2}\int_0^{1+\frac{1}{r^2}L^2 - \frac{2}{r}\cos\theta L} \frac{1}{2ru^{-3/2}}\, du = \frac{\lambda}{8\pi\epsilon_0 r^3}\left[\frac{-2}{\sqrt{1 + \frac{1}{r^2}x^2 - \frac{2}{r}\cos\theta x}}\right]_0^L$$

taylor series

$$= \boxed{\frac{\lambda}{8\pi\epsilon_0 r^3}\left(2 - \frac{2}{\sqrt{1 + \frac{1}{2}\left(\frac{2}{r}\cos\theta\right)L + \frac{1}{2}\left(\frac{3}{4}\left(\frac{4}{r^2}\cos^2\theta\right) - \frac{1}{r^2}\right)L^2}}\right)}$$

- 3b) (5 points) Find the monopole and (the magnitude of) the dipole moment for this arrangement of charge.

monopole = Q

dipole moment $\vec{p} = \vec{\delta} q = \vec{Q} \cdot \vec{r} =$

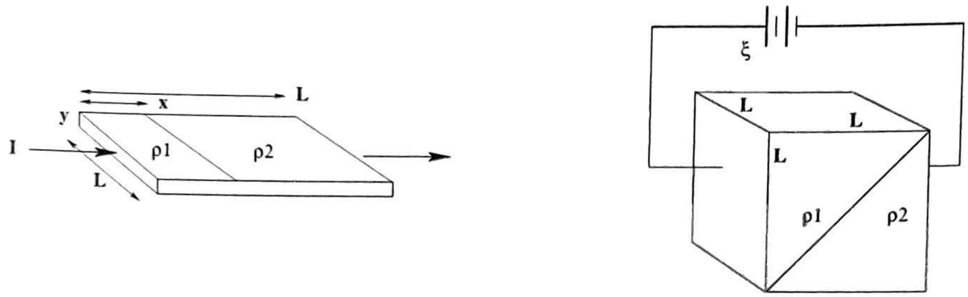
- 3c) (10 points) Find the radial (r), polar (θ) and azimuthal (ϕ) components of the electric field at a point described by r, θ and ϕ for which $r \gg L$.

$$E_r = -\frac{dV}{dr} \qquad E_\theta = \frac{-1}{r}\frac{dV}{d\theta} \qquad E_\phi = \frac{-1}{r\sin\theta}\frac{dV}{d\phi} = 0$$

$$E_r = +\frac{3\lambda}{8\pi\epsilon_0 r^4}\left(2 - \frac{2}{1+\left(\frac{1}{2r}\cos\theta\right)L+\left(\frac{3}{2r^2}\cos^2\theta-\frac{1}{2r^2}\right)L^2}\right)$$

$$+\frac{2\lambda}{8\pi\epsilon_0 r^3}\left(\frac{12\left(-\frac{\cos\theta}{r^2}+\left(-\frac{1}{2r^3}\cos^2\theta+\frac{1}{6r^3}\right)L^2\right)}{\left[1+\left(\frac{1}{r}\cos\theta\right)L+\frac{1}{2r^2}\left(3\cos^2\theta-1\right)L^2\right]^2}\right)$$

$$E_\theta = \frac{-2\lambda}{8\pi\epsilon_0 r^3}\left(\frac{-\frac{1}{r}\sin\theta L-\frac{3}{2r^2}\cos\theta\sin\theta L^2}{\left[1+\left(\frac{1}{r}\cos\theta\right)L+\frac{1}{2r^2}\left(3\cos^2\theta-1\right)L^2\right]^2}\right)$$



EXAM 4) In the diagram on the left, current flows into a thin square plate of constant cross-sectional area $L \times y$. It first encounters material of resistivity ρ_1 (having depth x) and then material of resistivity ρ_2. In the diagram on the right a battery of potential difference ξ drives current through a cubic resistor of edge-length L comprised of material of resistivity ρ_1 and ρ_2 arranged as shown in the diagram (the volume above the diagonal is filled with material ρ_1 and the volume below the diagonal is filled with material ρ_2).

- 4a) (5 points) First consider the diagram on the left. Derive the resistance and conductance the sheet presents to current flowing through the thin faces as shown.

$$C = \frac{\epsilon_0 A}{d} = \frac{\epsilon_0 y L}{L} = \epsilon_0 y$$

$$R = R_{\rho_1} + R_{\rho_2} = \frac{\rho_1 x}{yL} + \frac{\rho_2 (L-x)}{yL} = \frac{\rho_1 x + \rho_2 (L-x)}{yL}$$

- 4b) (15 points) Now consider the diagram on the right. Derive the resistance the cube presents to current flowing from the battery.

$$\int_0^R dR_{\rho_1} = \int_0^L \frac{\rho_1 dL'}{A} = \frac{\rho_1}{L} \int_0^L \frac{dL'}{L'} = \frac{\rho_1}{L}\left(\ln(L)\right) = \frac{\rho_1}{L}\ln(L)$$

$$\int_0^R dR_{\rho_2} = \int_0^L \frac{\rho_2 \, dL'}{A} = \frac{\rho_2}{L}\int_0^L \frac{dL'}{L'} = \frac{\rho_2}{L}\ln(L)$$

$$\therefore R = R_{\rho_1} + R_{\rho_2} = \frac{\rho_1}{L}\ln(L) + \frac{\rho_2}{L}\ln(L)$$

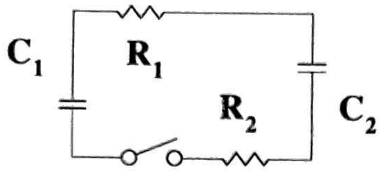
- 4c) (10 points) Derive the current density through the resistor as a function of height from bottom of the resistor.

$$J = \frac{E}{P} = \frac{I}{A} \qquad \mathcal{E} = I R_{eq} = I\left(\frac{P_1}{h} + \frac{P_2}{h}\right) \ln(b)$$

$$I = \frac{\mathcal{E}}{R_{req}} = \frac{\mathcal{E}}{\frac{\ln(b)}{h}(P_1 + P_2)}$$

$$\therefore J = \frac{I}{A} = \frac{I}{Lh} = \frac{\mathcal{E}}{L \ln(b)(P_1 + P_2)}$$

$$q_{1f} = \frac{v}{1 + \frac{C_2}{C_1}}$$



EXAM 5) In the circuit shown above, C_1 is given an an initial charge Q, C_2 is initially uncharged and the switch has been open for a long time. At $t = 0$ the switch is closed.

- 5a) (10 points) Derive the rate at which charge flows through the circuit after the switch is closed.

Kirchoff's First Law: $\frac{q}{C_1} - I(R_1+R_2) - \frac{(Q-q)}{C_2} = 0 \qquad I = -\frac{dq}{dt}$

$$\frac{q}{C_1} + \frac{q}{C_2} - \frac{Q}{C_2} + \frac{dq}{dt}(R_1+R_2) = 0 \qquad \frac{dq}{dt}(R_1+R_2) = \frac{-C_2q - C_1q + QC_1}{C_1C_2}$$

$$\int_Q^q \frac{dq}{QC_1-(C_2+C_1)q} = \int_0^t \frac{dt}{C_1C_2(R_1+R_2)} \Rightarrow \frac{\ln\left(QC_1-(C_2+C_1)q\right)}{-(C_2+C_1)}\Bigg|_Q^q = \frac{t}{C_1C_2(R_1+R_2)}$$

$$\frac{-1}{C_2+C_1}\left[\ln\left(\frac{QC_1-(C_2+C_1)q}{-QC_2}\right)\right] = \frac{t}{C_1C_2(R_1+R_2)} \Rightarrow \left(\frac{QC_1-(C_2+C_1)q}{-QC_2}\right)^{\frac{-1}{C_2+C_1}} = e^{\frac{t}{C_1C_2(R_1+R_2)}}$$

$$\left(\frac{-QC_2}{QC_1-(C_2+C_1)q}\right)^{\frac{-1}{C_2+C_1}} = e^{\frac{t}{C_1C_2(R_1+R_2)}} \qquad q = \frac{-1}{C_2 \cdot C_1}$$

- 5b) (5 points) How much charge is on each capacitor once the circuit reaches steady-state?

Conservation of charge: $q_{1i} + q_{2i} = q_{1f} + q_{2f}$

Steady state $\Rightarrow$ no current: $\Delta V_{AB} = \Delta V_{AB} \Rightarrow \frac{q_{1f}}{C_1} = \frac{q_{2f}}{C_2}$

Capacitor C_1

$$q_{2f} = \frac{q_{1f}C_2}{C_1}$$

$$Q + 0 = q_{1f} + \frac{C_2}{C_1}q_{1f}$$

$$q_{1f} = \frac{Q}{1+\frac{C_2}{C_1}}$$

Capacitor C_2

$$q_{1f} = \frac{q_{2f}C_1}{C_2}$$

$$Q + 0 = \frac{q_{2f}C_1}{C_2} + q_{2f}$$

$$q_{2f} = \frac{Q}{\frac{C_1}{C_2}+1}$$

- 5c) (5 points) How does the total energy stored in the circuit in steady-state compare to the total energy stored in the circuit initially (U_f/U_i)?

$$U_i = U_{1i} + U_{2i} = \frac{1}{2}\frac{Q^2}{C_1} + 0$$

$$U_f = U_{1f} + U_{2f} = \frac{1}{2}\frac{q_{1f}^2}{C_1} + \frac{1}{2}\frac{q_{2f}^2}{C_2} = \frac{1}{2}\left[\frac{Q^2}{C_1\left(1+\frac{C_2}{C_1}\right)^2} + \frac{Q^2}{C_2\left(1+\frac{C_1}{C_2}\right)^2}\right]$$

$$\therefore \frac{U_f}{U_i} = \frac{\cancel{\frac{1}{2}}\frac{\cancel{Q^2}}{C_1}}{\cancel{\frac{1}{2}}\left[\frac{\cancel{Q^2}}{C_1\left(1+\frac{C_2}{C_1}\right)^2} + \frac{\cancel{Q^2}}{C_2\left(1+\frac{C_1}{C_2}\right)^2}\right]} = \frac{1}{\frac{C_1}{C_1\left(1+\frac{C_2}{C_1}\right)^2} + \frac{C_1}{C_2\left(1+\frac{C_1}{C_2}\right)^2}} = \frac{1}{\left[\frac{1}{\left(1+\frac{C_2}{C_1}\right)^2} + \frac{C_1}{C_2\left(1+\frac{C_1}{C_2}\right)^2}\right]}$$

- 5d) (10 points) What fraction of the initial energy in the circuit was lost to R_1 ($\Delta U_{R1}/U_i$)?

$$P = \frac{\Delta V^2}{R_1} = I^2 R$$

$$\Delta U = \int_0^\infty I^2 R_1 \, dt = \Delta U_{R_1}$$

I would take the integral of part A, if I could find one):!

$$\frac{\Delta U_{R_1}}{U_i} = \frac{\Delta U}{\frac{1}{2}\frac{Q^2}{C_1}} = \frac{2C_1\int_0^\infty I^2 R_1 \, dt}{Q^2}$$