MIDTERM EXAM #2
Physics 1B
Instructor: Anton Bondarenko

Friday, November 17th, 2017
8:00 AM – 8:50 AM

Last nam

First nai

Univers

You will have **50 minutes** to complete this exam. One 3" x 5" index card and a calculator is permitted. **Notes, books, cell phones, and any other electronics are not allowed.** Please write your answer in the space below the problem. You must show your work to get full credit. Extra paper is attached at the back of the exam and more is available in the front of the room. If a problem seems confusing or ambiguous, please ask the proctor for clarifications.

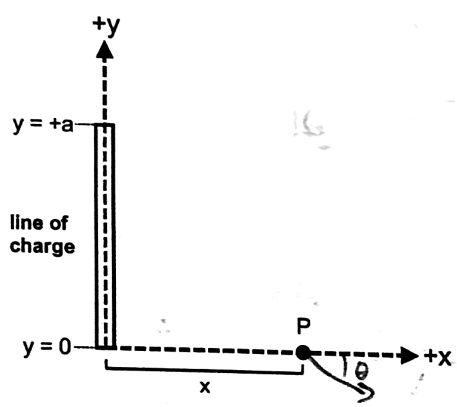
PLEASE DO NOT TURN PAGE UNTIL INSTRUCTED

Problem 1 (50 points total):

As shown in **Figure 1**, a line of charge is positioned along the y-axis between $y = 0$ and $y = +a$. The line of charge has a non-uniform positive linear charge density $\lambda = Cy$, where C is a positive constant and y is the co-ordinate along the y-axis.

Figure 1

$\lambda = Cy$



Part A (10 points): Write integral expressions for the x and y components of the electric field at point P, which is located along the x-axis at a distance x from the line of charge. Express your answer in terms of the given parameters and fundamental constants. **You do not have to evaluate the integrals in this part!**

$$E_x = \frac{1}{4\pi\epsilon_0} \int_0^{+a} \frac{Cy\, x\, dy}{(x^2+y^2)^{\frac{3}{2}}}$$ ✓

$$dE = \frac{1}{4\pi\epsilon_0} \frac{dQ}{r^2}$$

$$dE_x = \frac{1}{4\pi\epsilon_0} \frac{dQ}{r^2} \cos\theta$$

$$E_y = -\frac{1}{4\pi\epsilon_0} \int_0^{+a} \frac{Cy\, y\, dy}{(x^2+y^2)^{\frac{3}{2}}}$$

-1

$$dE_y = \frac{1}{4\pi\epsilon_0} \frac{dQ}{r^2} \sin\theta$$

$$\cos\theta = \frac{x}{(x^2+y^2)^{\frac{1}{2}}}$$

$$\sin\theta = \frac{y}{(x^2+y^2)^{\frac{1}{2}}}$$

9/10

Part B (10 points): Now, solve for the x component of the electric field in terms of the given parameters and fundamental constants by evaluating the corresponding integral from Part A. (Hint: you can evaluate the integral by U-substitution).

$$E_x = \frac{1}{4\pi\epsilon_0}\int_0^{+a} \frac{Cyx}{(x^2+y^2)^{\frac{3}{2}}}\,dy$$

$$= \frac{Cx}{4\pi\epsilon_0 2}\int_0^{+a} \frac{2y}{(x^2+y^2)^{\frac{3}{2}}}\,dy$$

$u = x^2 + y^2$

$\frac{du}{dy} = 2y$

$$= \frac{Cx}{8\pi\epsilon_0}\left[\frac{-2}{(x^2+y^2)^{\frac{1}{2}}}\right]_0^a$$

$$\boxed{= \frac{Cx}{8\pi\epsilon_0}\left[\frac{-2}{(x^2+a^2)^{\frac{1}{2}}} + \frac{2}{x}\right]}$$

simplificatio to check w/ part D

$$= \frac{2C}{4\pi\epsilon_0}\left[\frac{-x}{(x^2+a^2)^{\frac{1}{2}}} + 1\right] \checkmark$$

10/10

Part C (10 points): Solve for the electrostatic potential at point *P* in terms of the given parameters and fundamental constants by integrating over the line of charge. Assume that the potential is zero infinitely far away. (Hint: you can evaluate the integral by *U*-substitution).

$dV = \frac{1}{4\pi\epsilon_0} \cdot \frac{dQ}{r}$ $\quad V = \frac{1}{4\pi\epsilon_0} \int_0^{+a} \frac{cy}{(x^2+y^2)^{\frac{1}{2}}} dy$ $\qquad dQ = cy\,dy$

$V = \frac{c}{4\pi\epsilon_0 (2)} \left[\cancel{2} (x^2+y^2)^{\frac{1}{2}} \right]_0^{+a}$ $\qquad u = x^2+y^2$

$du = 2y$

$u^{-\frac{1}{2}} = 2u^{\frac{1}{2}}$

$$\boxed{V = \frac{c}{4\pi\epsilon_0} \left[(x^2+a^2)^{\frac{1}{2}} - (x^2)^{\frac{1}{2}} \right]}$$ ✓

10/10

Part D (10 points): From the expression for the electrostatic potential in Part C, obtain the *x* component of the electric field. (Your answer should agree with the answer to Part B).

$$E = -\nabla V$$

$$E_x = -\frac{\partial}{\partial x}(V)$$

$$V = \frac{c}{4\pi\epsilon_0}\left[(x^2+a^2)^{\frac{1}{2}} - x\right]$$

$$-\frac{\partial}{\partial x}(V) = -\frac{c}{4\pi\epsilon_0}\left[\frac{1}{2}(x^2+a^2)^{-\frac{1}{2}}\cdot 2x - 1\right]$$

$$= -\frac{c}{4\pi\epsilon_0}\left[\frac{x}{(x^2+a^2)^{\frac{1}{2}}} - 1\right]$$

$$\boxed{E_x = \frac{c}{4\pi\epsilon_0}\left[\frac{-x}{(x^2+a^2)^{\frac{1}{2}}} + 1\right]}$$

Part E (10 points): Show that if point P is much farther away along the x-axis than the length of the line of charge, the x component of the electric field is approximately that of a point charge Q at the origin, where Q is the total amount of charge contained on the line charge. You will need the following Taylor series approximation:

$$\frac{1}{\sqrt{1+c}} \approx 1 - \frac{c}{2} \quad (\text{for } c << 1)$$

$$E_x = \frac{C}{4\pi\epsilon_0}\left[\frac{-x}{(x^2+a^2)^{\frac{1}{2}}} + 1\right]$$

$$E_x = \frac{C}{4\pi\epsilon_0}\left[\frac{-1}{\left(1+\frac{a^2}{x^2}\right)^{\frac{1}{2}}} + 1\right]$$

$$\lim_{x\to\infty} \frac{C}{4\pi\epsilon_0}\left[\frac{-1}{\left(1+\frac{a^2}{x^2}\right)^{\frac{1}{2}}} + 1\right]$$

use Taylor Series app

$$= \frac{C}{4\pi\epsilon_0}\left[-\left(\cancel{1} - \frac{\frac{a^2}{x^2}}{2}\right) + \cancel{1}\right]$$

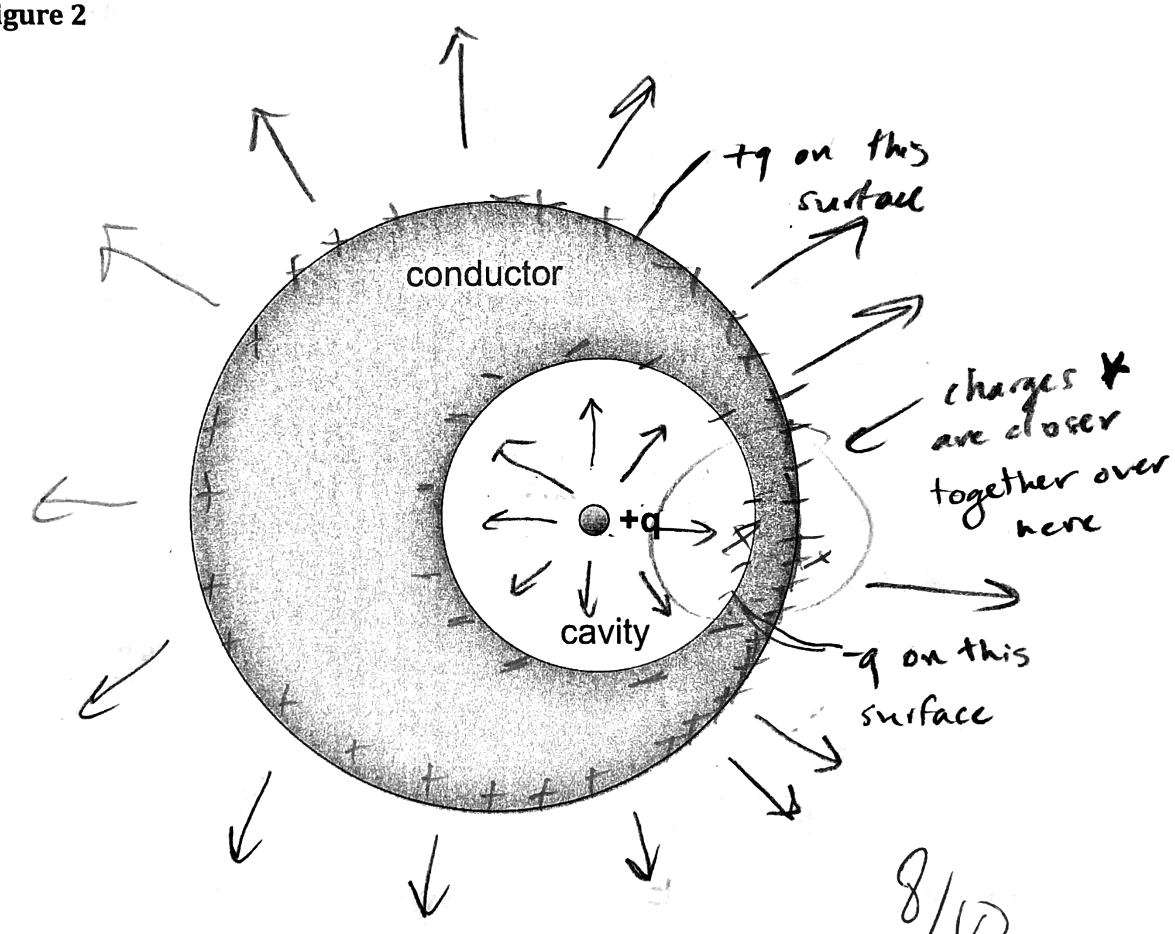
$$= \frac{C}{4\pi\epsilon_0}\frac{a^2}{2x^2} = \boxed{\frac{C a^2}{8\pi\epsilon_0 x^2}}$$

Since $Q = \frac{Ca^2}{2}$ this agrees w/ $\frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}$ or the electric field due to a point charge

Problem 2 (20 points total):

As shown in **Figure 2**, an initially uncharged spherical conductor has an off-center spherical cavity. In the center of the cavity is a positive point charge *+q.*

Figure 2



Part A (10 points): In the figure above, sketch (1) the distribution of charge on the conductor and (2) the electric field lines everywhere. Use "+" signs for positive charges and "-" signs for negative charges.

No electric field lines inside conductor

Part B (10 points): How much charge, if any, would you need to add to the conductor so that the electric field outside of the conductor (that is, outside of the outer surface) vanishes? Briefly justify your answer.

~~You w~~ would need to add a net charge of $-q$ because then $Q_{enclosed}$ would be equal to 0 and by Gauss' Law the electric field would be 0

$$\oint \vec{E} \cdot d\vec{A} = \frac{Q_{encl}}{\epsilon_0}$$

if $Q_{encl} = 0$ then

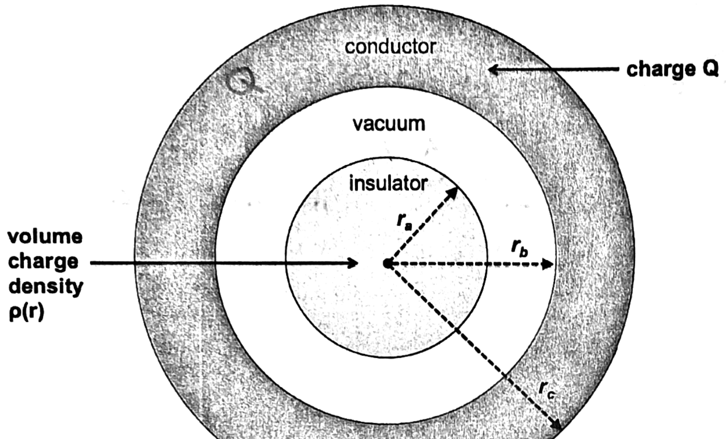
$$E(4\pi r^2) = \frac{0}{\epsilon_0}$$

$$\vec{E} = \vec{0}$$

Problem 3 (30 points total):

As shown in **Figure 3**, a solid insulating sphere of radius r_a has a radially dependent, positive volume charge density $\rho(r) = Cr$, where C is a positive constant and r is the radial coordinate. The insulating sphere is centered within a conducting shell of inner radius r_b and outer radius r_c. The conducting shell has a net positive charge Q.

Figure 3



Part A (10 points): Determine (1) the net charge on the inner surface of the conducting shell and (2) the net charge on the outer surface of the conducting shell. Express your answers in terms of the given parameters and fundamental constants.

(1) Insulator total charge $= Cr\left(\frac{4}{3}\pi r_a^3\right) = C\frac{4}{3}\pi r_a^4$

So, net charge on inner surface of conductor must cancel this so, $\boxed{-C\frac{4}{3}\pi r_a^4 = \text{charge on inner surface of conductor}}$

(2) Charge on conductor must add to Q

-8

$Q = \text{charge on outer shell} + -C\frac{4}{3}\pi r_a^4$

$\Rightarrow \boxed{\text{charge on outer shell} = Q + C\frac{4}{3}\pi r_a^4}$

Part B (10 points): Determine the magnitude and direction of the electric field (1) in the vacuum region ($r_a < r < r_b$), (2) inside the conducting shell ($r_b < r < r_c$), and (3) outside the conducting shell ($r > r_c$). Express your answers in terms of the given parameters and fundamental constants.

(1) $\oint \vec{E} \cdot d\vec{A} = \dfrac{Q_{enclosed}}{\epsilon_0}$ $\qquad Q_{enc} = Cr\left(\frac{4}{3}\pi r_a^3\right) = C\frac{4}{3}\pi r_a^4$

$(4\pi r^2)(E) = \dfrac{C\frac{4}{3}\pi r_a^4}{\epsilon_0}$

$E = \dfrac{C\frac{4}{3}\pi r_a^4}{4\pi r^2 \epsilon_0} = \boxed{\dfrac{C r_a^4}{3r^2\epsilon_0}}$ in the radially out ward direction

(2) Q enclosed $= 0 \therefore \boxed{E = 0}$

(3) Q enclosed $= C\frac{4}{3}\pi r_a^4 + Q$

$\oint \vec{E} \cdot d\vec{A} = \dfrac{Q}{\epsilon_0}$

$(4\pi r^2)\, E = C\frac{4}{3}\pi r_a^4 + Q$

$\boxed{E = \dfrac{C\frac{4}{3}\pi r_a^4 + Q}{4\pi r^2}}$ in the radially outward direction

Part C (10 points): Taking the electrostatic potential to be zero infinitely far away, calculate the electrostatic potential (1) outside the conducting shell ($r > r_c$) and (2) inside the conducting shell ($r_b < r < r_c$). Express you answers in terms of the given parameters and fundamental constants.

(1) $V_A - V_B = \int_r^{\infty} \vec{E} \cdot d\ell$

$V_A - 0 = \int_r^{\infty} \frac{C \frac{4}{3} \pi r_a^4 + Q}{4\pi r^2}$

$$\boxed{V_A = - \frac{C \frac{4}{3} \pi r_a^4 + Q}{4\pi r}}$$

-1

(2) $V_A - V_B = \int_{R_c}^{\infty} \frac{C \frac{4}{3} \pi r_a^4 + Q}{4\pi r^2} + \int_r^{R_c} 0$

$$\boxed{V_A = - \frac{C \frac{4}{3} \pi r_a^4 + Q}{4\pi R_c}} + 0$$

SCORING

Problem 1:
Part A: 9 / 10
Part B: 10 / 10
Part C: 10 / 10
Part D: 10 / 10
Part E: 10 / 10
Total: 49 / 50

Problem 2:
Part A: 8 / 10
Part B: 10 / 10
Total: 18 / 20

Problem 3:
Part A: 2 / 10
Part B: 10 / 10
Part C: 9 / 10
Total: 21 / 30

Total Midterm #2 Score 88 / 100