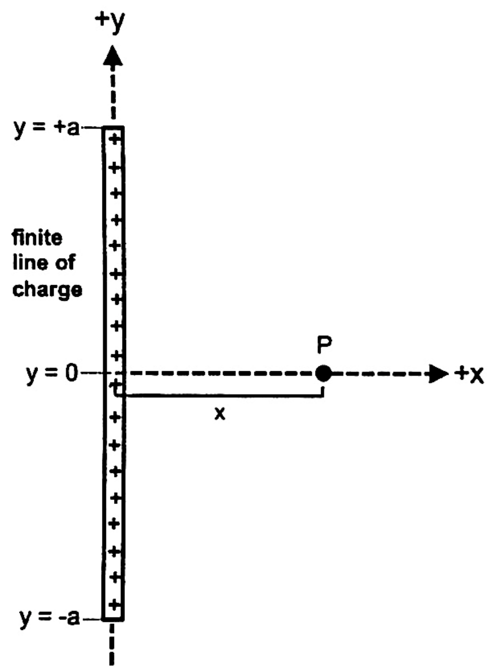
Problem 1 (40 points total):

Figure 1 below shows a finite line of charge positioned along the y-axis between $y = -a$ and $y = +a$. The line of charge has a uniform positive linear charge density λ

Figure 1



Part A (10 points): (1) Derive integral expressions for the *x* and *y* components of the electric field at point *P*, which is located along the x-axis at a distance *x* from the line of charge. (2) Then, evaluate the integrals to obtain the magnitude and direction of the electric field at point *P*. Express your answer in terms of the given parameters and fundamental constants. You will need the following integral:

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$$\int \frac{dy}{(c^2+y^2)^{3/2}} = \frac{y}{c^2\sqrt{c^2+y^2}}$$

1) $\lambda = \frac{Q}{2a}$

$E_x = \vec{E}\cos\theta$ $\quad E = \frac{dq}{4\pi\varepsilon_0 r^2}$ $\quad dq = \lambda dy$

$E_y = \vec{E}\sin\theta$ $\quad r = \sqrt{x^2+y^2}$

$\cos\theta = \frac{x}{\sqrt{x^2+y^2}}$

$\sin\theta = \frac{y}{\sqrt{x^2+y^2}}$

$$E_x = \int_{-a}^{a} \frac{dq}{4\pi\varepsilon_0(x^2+y^2)}\cdot\cos\theta = \int_{-a}^{a}\frac{x\,dq}{4\pi\varepsilon_0(x^2+y^2)^{3/2}} \rightarrow \int_{-a}^{a}\frac{x\lambda dy}{4\pi\varepsilon_0(x^2+y^2)^{3/2}} = E_x$$

$$E_y = \int_{-a}^{a}\frac{dq}{4\pi\varepsilon_0(x^2+y^2)}\cdot\sin\theta = \int_{-a}^{a}\frac{y\,dq}{4\pi\varepsilon_0(x^2+y^2)^{3/2}} \rightarrow \int_{-a}^{a}\frac{y\lambda dy}{4\pi\varepsilon_0(x^2+y^2)^{3/2}} = E_y$$

2) $$E_x = \frac{\lambda x}{4\pi\varepsilon_0}\int_{-a}^{a}\frac{dy}{(x^2+y^2)^{3/2}} = \frac{\lambda x}{4\pi\varepsilon_0}\cdot\frac{y}{x^2\sqrt{x^2+y^2}}\Big|_{-a}^{a}$$

$$\frac{\lambda x}{4\pi\varepsilon_0 x^2}\left(\frac{a}{\sqrt{x^2+a^2}} - \frac{-a}{\sqrt{x^2+a^2}}\right) = \frac{2\lambda x a}{4\pi\varepsilon_0 x^2\sqrt{x^2+a^2}}$$

$$E_y = \frac{\lambda x}{4\pi\varepsilon_0}\int_{-a}^{a}\frac{x\,dy}{(x^2+y^2)^{3/2}} \qquad u = x^2+y^2,\ du = 2y \rightarrow \frac{\lambda x}{4\pi\varepsilon_0}\int_{-a}^{a}\frac{du}{2(u)^{3/2}} = \frac{\lambda x}{4\pi\varepsilon_0}\left(-u^{-\frac{1}{2}}\right|_{-a}^{a}$$

$$\frac{\lambda x}{4\pi\varepsilon_0}\left(-\sqrt{x^2+a^2}\right|_{-a}^{a}$$

$$\frac{\lambda x}{4\pi\varepsilon_0}\cdot\left(-\sqrt{2a^2}+\sqrt{2a^2}\right)$$

$= 0$

by symmetry E_y is 0

$$\boxed{@\text{ point } P:\ \vec{E} = \frac{\cancel{2\lambda a}\ \lambda a}{2\pi\varepsilon_0 x\sqrt{x^2+a^2}} \uparrow}$$

Part B (10 points): Now, determine the magnitude and direction of the electric field at point *P* due to an infinitely long line charge by evaluating the expression from Part B in the limit *a* >> *x*.

10

$$\vec{E} = \frac{\lambda a}{2\pi\varepsilon_0 x\sqrt{x^2+a^2}}\,\hat{i}$$

$a \gg x$:

$$\frac{\lambda a}{2\pi\varepsilon_0 x}\cdot\frac{1}{\sqrt{x^2+a^2}}$$

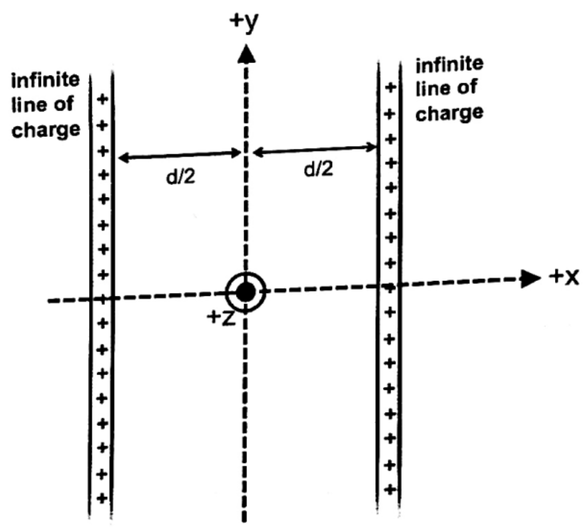
$$\sqrt{x^2+a^2} \approx \sqrt{a^2}$$

$$\frac{\lambda a}{2\pi\varepsilon_0 x}\cdot\frac{1}{a}$$

$$= \boxed{\frac{\lambda}{2\pi\varepsilon_0 x}\,\hat{i}}$$

Figure 2 below shows two identical infinite line charges of positive linear charge density λ that are placed in the xy plane and aligned along the y-axis. The two line charges are separated by a distance d, with each line charge positioned at a distance $d/2$ from the y-axis.

Figure 2



Part C (10 points): Determine the magnitude and direction of the electric field as a function of x in the range $-d/2 < x < +d/2$ in the xy plane (i.e., between the two line charges). Hint: use the result from Part B and the principle of superposition.

10

$$\vec{E} = \frac{\lambda}{2\pi\varepsilon_0 x}\hat{i}$$

from line @ $x = -\frac{d}{2}$

$$\frac{\lambda}{2\pi\varepsilon_0 r}$$

from line @ $x = \frac{d}{2}$

$$\frac{\lambda}{2\pi\varepsilon_0 x}$$

$$\frac{\lambda}{2\pi\varepsilon_0 r_1} - \frac{\lambda}{2\pi\varepsilon_0 r_2}$$

$$\frac{\lambda}{2\pi\varepsilon_0}\left[\frac{1}{r_1} - \frac{1}{r_2}\right]\hat{i}$$

$r_1 = \left(x + \frac{d}{2}\right)$

$r_2 = -\left(x - \frac{d}{2}\right)$

$$\frac{\lambda}{2\pi\varepsilon_0}\left[\frac{1}{x+\frac{d}{2}} - \frac{1}{\frac{d}{2}-x}\right]\hat{i}$$

Part D (10 points): A positive point charge q_0 that has a mass m_0 is placed between the two line charges in the xy plane at a very small displacement from the origin along the x-axis, such that $|x| << d$. (1) Show that the resulting motion of the charge will be approximately simple harmonic motion, and (2) derive an expression for the frequency of oscillations. Express your answer in terms of the given parameters and fundamental constants. You will need the following Taylor series approximations:

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$$\frac{1}{1+c} \approx 1 - c \quad (\text{for } c << 1)$$

$$\frac{1}{1-c} \approx 1 + c \quad (\text{for } c << 1)$$

shm: $F = -k\Delta x$

$$F_{e_1} + F_{e_2} = ma$$

$$q_0\left(E_{e_1} + E_{e_2}\right) = \frac{q_0 \lambda}{2\pi\varepsilon_0} \underbrace{\left[\frac{1}{x+\frac{d}{2}} - \frac{1}{\frac{d}{2}-x}\right]}_{\Delta x} = ma \quad \approx \text{shm}$$

$$\frac{q_0 \lambda}{2\pi\varepsilon_0}\left[\frac{1}{x+\frac{d}{2}} - \frac{1}{\frac{d}{2}-x}\right] = ma$$

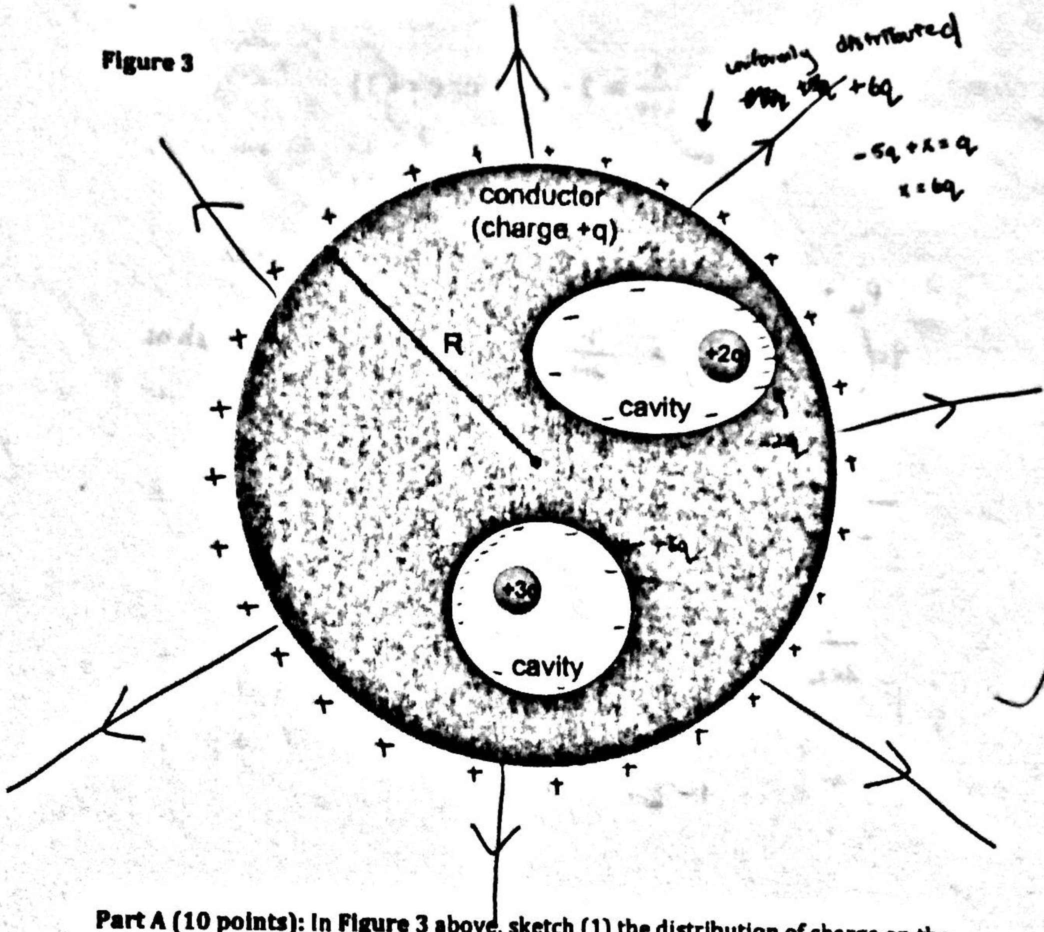
$$\frac{q_0 \lambda}{2\pi\varepsilon_0 m}\left[\frac{1}{x+\frac{d}{2}} - \frac{1}{\frac{d}{2}-x}\right] = a$$

1

Problem 2 (30 points total):

As shown in **Figure 3** below, a spherical conductor of radius *R* holds a net positive charge *+q*. The conductor also contains two internal cavities. One of the cavities contains a positive point charge *+2q*, and the other cavity contains a positive point charge *+3q*. For this problem, assume the space in the cavities and outside of the spherical conductor is vacuum.

Figure 3



Part A (10 points): In **Figure 3** above, sketch (1) the distribution of charge on the conductor and (2) the electric field lines outside of the conductor (*r* > *R*). Use "+" signs for positive charge and "-" signs for negative charge. The spacing between the signs should represent the relative charge density.

Part B (10 points): Determine the magnitude and direction of the electric field as a function of radial distance *r* from the conductor center (1) outside of the conductor (*r* > *R*) and (2) inside of the material of the conductor. Express your answer in terms of the given parameters and fundamental constants.

$r > R$:

$$\oint \vec{E} \cdot dA = \frac{q_{enc}}{\varepsilon_0}$$

By symmetry, $\oint \vec{E}_{net} \cdot d\vec{A} = E_{net}(4\pi r^2)$

$$q_{enc} = 6q$$

$$\vec{E}_{net} = \frac{6q}{4\pi\varepsilon_0 r^2}\hat{r} = \boxed{\frac{3q}{2\pi\varepsilon_0 r^2}\hat{r}}$$

$r < R$:

inside the conductor, $\vec{E}$ is 0 ~~by definit.~~

because it is a conductor

Part C (10 points): Taking the electrostatic potential to be zero infinitely far away, determine the potential as a function of radial distance *r* from the conductor center (1) outside of the conductor *(r > R)* and (2) inside of the material of the conductor. Express your answer in terms of the given parameters and fundamental constants.

1) $\Delta V = \int \vec{E} \cdot d\vec{\ell}$

$V_A - V_B = \int_A^B \vec{E} \cdot d\hat{\ell}$

$V_B = 0$, if B is ∞

$E = \frac{6q}{4\pi\varepsilon_0 r^2}\hat{r}$

$V_A = \int_A^\infty \vec{E} \cdot d\vec{\ell}$

assume $d\vec{\ell} \parallel \hat{A}$.

$d\vec{\ell} = |d\vec{\ell}|dr$

so $\hat{r} \cdot |d\vec{\ell}|dr$ is 1

$\int_A^\infty \frac{6q}{4\pi\varepsilon_0 r^2} dr$

$\frac{6q}{4\pi\varepsilon_0} \cdot -\frac{1}{r}\Big|_A^\infty$

$\frac{6q}{4\pi\varepsilon_0}\left(-\frac{1}{\infty} + \frac{1}{A}\right) \rightarrow \frac{6q}{4\pi\varepsilon_0 A}$ where A is r

$V = \frac{6q}{4\pi\varepsilon_0 r} = \boxed{\frac{3q}{2\pi\varepsilon_0 r}}$

2) inside the conductor:

$V_A - V_B = \int_r^\infty \vec{E} \cdot d\vec{\ell} = \underbrace{\int_r^R \vec{E} \cdot d\vec{\ell}}_{} + \int_R^\infty \vec{E} \cdot d\vec{\ell}$

0 since $\vec{E}$ is 0 within the conductor

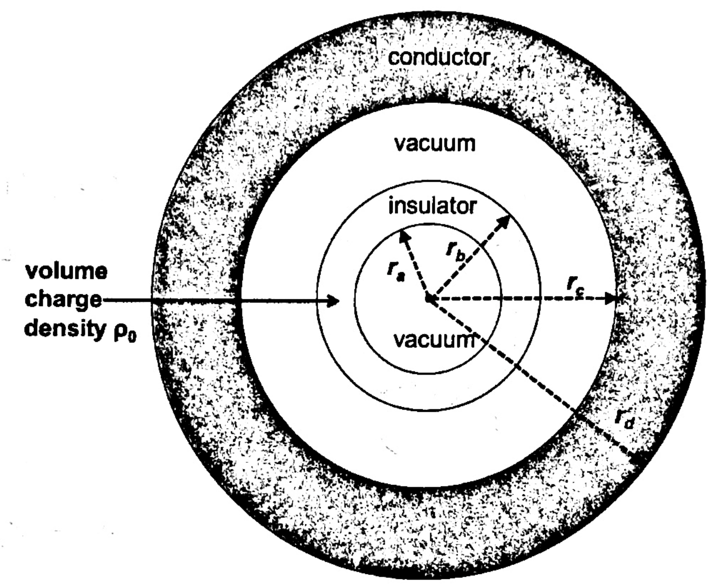
$\frac{6q}{4\pi\varepsilon_0 R}$

$V = \frac{6q}{4\pi\varepsilon_0 R} = \boxed{\frac{3q}{4\pi\varepsilon_0 R}}$

Problem 3 (30 points total):

Figure 4 shows the cross-section of an infinitely long insulating cylindrical shell of inner radius r_a and outer radius r_b that has a uniform positive volume charge density ρ_0. The insulating shell is centered within an infinitely long uncharged conducting cylindrical shell of inner radius r_c and outer radius r_d.

Figure 4



Part A (10 points): Determine the surface charge density on (1) the inner surface of the conducting shell and (2) the outer surface of the conducting shell. Express your answers in terms of the given parameters and fundamental constants.

1) inside of conductor = - insulating layer

$$Q_{ins} = \int_{r_a}^{r_b} \rho_0 dV$$

$\cancel{dV = 4\pi r^2 dr}$ $\quad dV = 2\pi r L dr$

$\cancel{\int_{r_a}^{r_b} 4\pi r^2 \rho_0 dr}$ $\quad \int_{r_a}^{r_b} \rho_0 2\pi r L dr$

$\cancel{4\pi\rho_0 \int_{r_a}^{r_b} r^2 dr}$ $\quad Q = \rho_0 (\pi r^2 L) \Big|_{r_a}^{r_b}$

$\cancel{4\pi\rho_0 \left(\frac{1}{3} r^3 \Big|_{r_a}^{r_b}\right)}$ $\quad Q = \rho_0 L(\pi)(r_b^2 - r_a^2)$

$\cancel{Q_{ins} = 4\pi\rho_0 \frac{1}{3}(r_b^3 - r_a^3)}$ $\quad Q_{insu} = \rho_0 L\pi (r_b^2 - r_a^2)$

$$-\rho_0 L\pi (r_b^2 - r_a^2)$$

inside of ~~conductor~~ conducting shell has total charge: $\cancel{-\frac{4}{3}\pi\rho_0 (r_b^3 - r_a^3)}$

the surface charge density is $\frac{Q}{A} = \frac{Q}{\pi r^2} \rightarrow \cancel{-\frac{\frac{4}{3}\pi\rho_0 (r_b^3 - r_a^3)}{\pi r_c^2}}$ $\quad \frac{-\rho_0 L\pi (r_b^2 - r_a^2)}{\pi r^2 L}$

1) $\boxed{\sigma = \frac{-\rho_0 \pi (r_b^2 - r_a^2)}{\pi r_c}}$

2) $Q_{out} = Q_{ins} = \frac{4}{3}\pi\rho_0 (r_b^3 - r_a^3)$

~~surface ch~~

$\sigma = \frac{Q}{A} \rightarrow \cancel{\frac{\frac{4}{3}\pi\rho_0 (r_b^3 - r_a^3)}{\pi r_d^2}}$

$A = \pi r^2 L$

2) $\boxed{\sigma = \frac{\rho_0 \pi (r_b^2 - r_a^2)}{\pi r_d}}$

Part B (10 points): Determine the magnitude and direction of the electric field as a function of radial distance r from the center (1) inside the insulating shell ($r_a < r < r_b$) and (2) in the vacuum region between the insulating shell and the conducting shell ($r_b < r < r_c$). Express your answers in terms of the given parameters and fundamental constants.

1) $r_a < r < r_b$

$$\int \vec{E}\, dA = \frac{q_{enc}}{\varepsilon_0}$$

By symmetry, $\vec{E}\, dA = E(2\pi r L)$ for both parts

$$q_{enc}: \int_{r_a}^{r} \rho_0\, dV = \rho_0 V \Big|_{r_a}^{r} = \rho_0 (\pi r^2 L) \Big|_{r_a}^{r}$$

$$E(2\pi r L) = \frac{\rho_0 (\pi r^2 L - \pi r_a^2 L)}{\varepsilon_0}$$

$$E = \frac{\rho_0 (r^2 - r_a^2)}{2 \varepsilon_0 r} \hat{r} = \boxed{\frac{\rho_0 (r^2 - r_a^2)}{2 \varepsilon_0 r} \hat{r}}$$

2) $r_b < r < r_c$: $\int_{r_a}^{r_b} \rho_0\, dV = \rho_0 V \Big|_{r_a}^{r_b}$

$$\vec{E}(2\pi r L) = \frac{\rho_0 \pi L (r_b^2 - r_a^2)}{\varepsilon_0}$$

$$\boxed{\vec{E} = \frac{\rho_0 (r_b^2 - r_a^2)}{2 \varepsilon_0 r} \hat{r}}$$

Part C (10 points): Calculate the voltage (i.e., the magnitude of the potential difference) between the outer surface of the insulating shell ($r = r_b$) and the inner surface of the conducting shell ($r = r_c$). Express your answer in terms of the given parameters and fundamental constants.

$$V_{r_c} = \int_r^\infty \vec{E}\cdot d\vec{l} = \underbrace{\int_r^{r_d} \vec{E}\cdot d\vec{l}}_{0,\ \text{inside conductor}} + \underbrace{\int_{r_d}^\infty \vec{E}\cdot d\vec{l}}_{\int_{r_d}^\infty \frac{\rho_0 (r_b^2 - r_a^2)}{2\varepsilon_0 r} dr}$$

$$\frac{\rho_0 (r_b^2 - r_a^2)}{2\varepsilon_0} \left[\ln(\infty) - \ln(r_d)\right]$$

$$V_{r_b} = \int_r^\infty \vec{E}\cdot d\vec{l} = \int_r^{r_c} \vec{E}\cdot d\vec{l} + \underbrace{\int_{r_c}^{r_d} \vec{E}\cdot d\vec{l}}_{0} + \int_{r_d}^\infty \vec{E}\cdot d\vec{l}$$

$$V_{r_b} - V_{r_c} = \int_{r_b}^{r_c} \vec{E}\cdot d\vec{l} + \cancel{\int_{r_d}^\infty \vec{E}\cdot d\vec{l} - \int_{r_d}^\infty \vec{E}\cdot d\vec{l}}$$

$$\int_{r_b}^{r_c} \frac{\rho_0 (r_b^2 - r_a^2)}{2\varepsilon_0 r} dr$$

$$\frac{\rho_0 (r_b^2 - r_a^2)}{2\varepsilon_0} (\ln r)\Big|_{r_b}^{r_c}$$

$$V_{r_b} - V_{r_c} = \frac{\rho_0 (r_b^2 - r_a^2)}{2\varepsilon_0} \left[\ln(r_c) - \ln(r_b)\right]$$