MIDTERM EXAM #2
Physics 1B Lecture 4
Instructor: Anton Bondarenko

Tuesday, May 22nd, 2018
10:00 AM – 10:50 AM

Last name: Pugh

First name: Spencer

University ID: 605 006 452

You will have **50 minutes** to complete this exam. One standard 3" x 5" index card and a calculator are permitted. Notes, books, cell phones, and any other electronics are not allowed. **You must complete the exam using a blue or black ink pen. Exams completed in pencil will not receive credit.**

Please write your answer in the space below the problem. **You must write legibly and demonstrate your reasoning to get full credit.** For quantitative problems, always express your final answer in terms of the variables given in the problem unless otherwise stated. For clarity, please draw a box around your final answer.

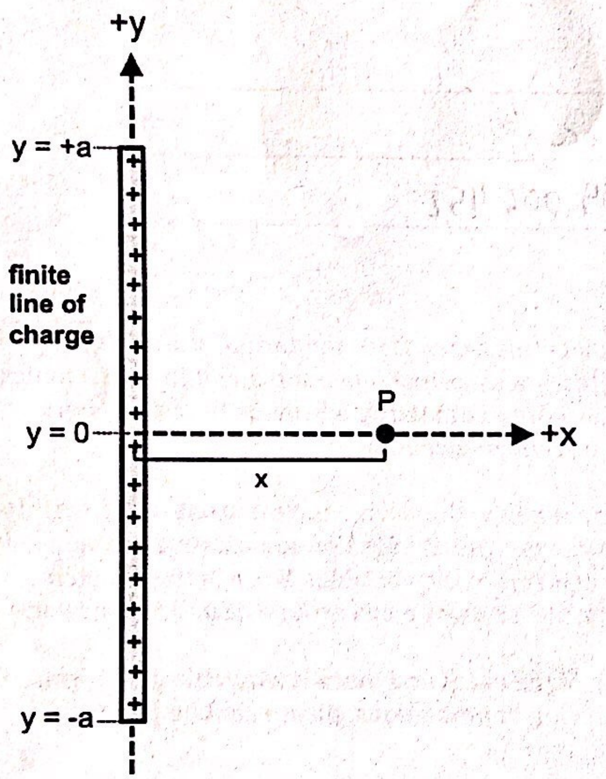
Extra paper is attached at the back of the exam and more is available in the front of the room. If a problem seems confusing or ambiguous, please ask the proctor for clarifications.

DO NOT TURN PAGE UNTIL INSTRUCTED

Problem 1 (40 points total):

Figure 1 below shows a finite line of charge positioned along the y-axis between $y = -a$ and $y = +a$. The line of charge has a uniform positive linear charge density λ.

Figure 1



Part A (10 points): (1) Derive integral expressions for the *x* and *y* components of the electric field at point *P*, which is located along the x-axis at a distance *x* from the line of charge. (2) Then, evaluate the integrals to obtain the magnitude and direction of the electric field at point *P*. Express your answer in terms of the given parameters and fundamental constants. You will need the following integral:

$$\int \frac{dy}{(c^2+y^2)^{3/2}} = \frac{y}{c^2\sqrt{c^2+y^2}}$$

(1) $\boxed{E_y = 0}$ by symmetry $\quad r^2 = x^2+y^2 \quad \cos\theta = \frac{x}{\sqrt{x^2+y^2}}$

$$E_x = \int dE_x = \int \frac{1}{4\pi\varepsilon_0}\frac{dq}{r^2}\cos\theta = \frac{1}{4\pi\varepsilon_0}\int_{-a}^{a}\frac{x\lambda\, dy}{(x^2+y^2)^{3/2}} = \frac{\lambda x}{4\pi\varepsilon_0}\int_{-a}^{a}\frac{dy}{(x^2+y^2)^{3/2}}$$

$$= \frac{\lambda x}{4\pi\varepsilon_0}\left[\frac{y}{x^2\sqrt{x^2+y^2}}\right]_{-a}^{a} = \frac{\lambda x}{4\pi\varepsilon_0}\left[\frac{a}{x^2\sqrt{x^2+a^2}} - \frac{-a}{x^2\sqrt{x^2+a^2}}\right] = \frac{2a\lambda x}{4\pi\varepsilon_0 x^2\sqrt{x^2+a^2}} = \frac{a\lambda}{2\pi\varepsilon_0 x\sqrt{x^2+a^2}}$$

$$\boxed{E_x = \frac{a\lambda}{2\pi\varepsilon_0 x\sqrt{x^2+a^2}}}$$

(2) Since $E_y = 0$, $|\vec{E}| = E_x = \frac{a\lambda}{2\pi\varepsilon_0 x\sqrt{x^2+a^2}}$

direction = $+x$ direction (charge is positive) a.k.a. the $+\hat{i}$ direction if $x > 0$
$-x$ direction (charge is positive) a.k.a the $-\hat{i}$ direction if $x < 0$

$$\boxed{\vec{E} = E_x\hat{i} = \frac{a\lambda}{2\pi\varepsilon_0 x\sqrt{x^2+a^2}}\hat{i}}$$

direction is <u>away</u> from y-axis

Part B (10 points): Now, determine the magnitude and direction of the electric field at point *P* due to an infinitely long line charge by evaluating the expression from Part [redacted] in the limit *a* >> *x*.

part A: $\vec{E} = \frac{a\lambda}{2\pi\varepsilon_0 x\sqrt{x^2+a^2}}\hat{i}$

$a >> x$: $\sqrt{x^2+a^2} \approx \sqrt{a^2} = a$

$$\vec{E} \approx \frac{\cancel{a}\lambda}{2\pi\varepsilon_0 \cancel{a} x}\hat{i} = \frac{\lambda}{2\pi\varepsilon_0 x}\hat{i}$$

$$\vec{E} = \frac{\lambda}{2\pi\varepsilon_0 x}\hat{i}$$

$$|\vec{E}| = \frac{\lambda}{2\pi\varepsilon_0 x}$$

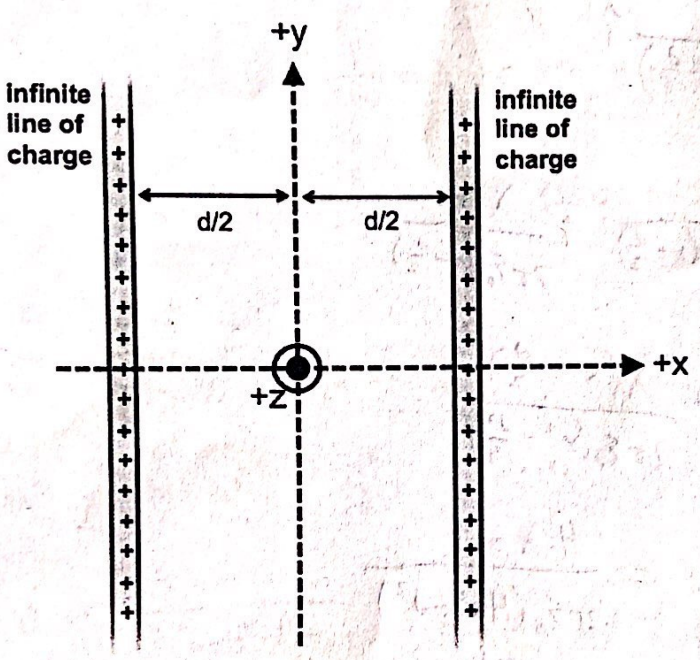
direction: away from y-axis

direction: $+\hat{i}$ if $x > 0$

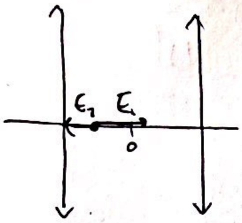
$-\hat{i}$ if $x < 0$

Figure 2 below shows two identical infinite line charges of positive linear charge density λ that are placed in the xy plane and aligned along the y-axis. The two line charges are separated by a distance d, with each line charge positioned at a distance $d/2$ from the y-axis.

Figure 2



Part C (10 points): Determine the magnitude and direction of the electric field as a function of x in the range $-d/2 < x < +d/2$ in the xy plane (i.e., between the two line charges). Hint: use the result from Part B and the principle of superposition.



$$\vec{E}_1 = \frac{\lambda}{2\pi\varepsilon_0\left(x+\frac{d}{2}\right)}\,\hat{\imath}$$

$$\vec{E}_2 = \frac{\lambda}{2\pi\varepsilon_0\left(x-\frac{d}{2}\right)}\,\hat{\imath}$$

$$\vec{E}_{tot} = \vec{E}_1 + \vec{E}_2 = \frac{\lambda}{2\pi\varepsilon_0\left(x+\frac{d}{2}\right)}\hat{\imath} + \frac{\lambda}{2\pi\varepsilon_0\left(x-\frac{d}{2}\right)}\hat{\imath}$$

$$= \frac{\lambda\left(x-\frac{d}{2}\right)}{2\pi\varepsilon_0\left(x^2-\frac{d^2}{4}\right)}\hat{\imath} + \frac{\lambda\left(x+\frac{d}{2}\right)}{2\pi\varepsilon_0\left(x^2-\frac{d^2}{4}\right)}\hat{\imath}$$

$$= \frac{2\lambda x\,\hat{\imath}\cdot 4}{4\cdot 2\pi\varepsilon_0\left(x^2-\frac{d^2}{4}\right)} = \frac{4\lambda x\,\hat{\imath}}{\pi\varepsilon_0\left(4x^2-d^2\right)}$$

$$\boxed{\vec{E} = \frac{4\lambda x\,\hat{\imath}}{\pi\varepsilon_0\left(4x^2-d^2\right)}}$$

magnitude: $\left|\frac{4\lambda x}{\pi\varepsilon_0\left(4x^2-d^2\right)}\right|$

direction: toward $x=0$

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Part D (10 points): A positive point charge q_0 that has a mass m_0 is placed between the two line charges in the xy plane at a very small displacement from the origin along the x-axis, such that $|x| << d$. (1) Show that the resulting motion of the charge will be approximately simple harmonic motion, and (2) derive an expression for the frequency of oscillations. Express your answer in terms of the given parameters and fundamental constants. You will need the following Taylor series approximations:

$$\frac{1}{1+c} \approx 1 - c \quad (\text{for } c << 1)$$

$$\frac{1}{1-c} \approx 1 + c \quad (\text{for } c << 1)$$

(1) $\vec{F} = \dfrac{4\lambda x q_0 \hat{i}}{\pi \varepsilon_0 (4x^2 - d^2)} = m_0 \vec{a}$

simple harmonic motion if $\dfrac{d^2}{dt^2}x = -\omega^2 x$ aka $a \sim -x$

$a = \dfrac{4\lambda x}{\pi \varepsilon_0 (4x^2 - d^2) m_0} = \dfrac{4\lambda}{\pi \varepsilon_0 m_0}\left(\dfrac{x}{4x^2 - d^2}\right) \approx \dfrac{-4\lambda}{\pi \varepsilon_0 m_0 d^2} x$ if $|x| << d$

bc $x^2 \approx 0$

so it is approx SHM

factor of 2

(2) $\omega^2 = \dfrac{4\lambda}{\pi \varepsilon_0 m_0 d^2} \rightarrow \omega = \sqrt{\dfrac{4\lambda}{\pi \varepsilon_0 m_0 d^2}}$

missing

$f = \dfrac{\omega}{2\pi} = \dfrac{1}{2\pi}\sqrt{\dfrac{4\lambda}{\pi \varepsilon_0 m_0 d^2}} = \sqrt{\dfrac{\cancel{4}\lambda}{\cancel{4}\pi^3 m_0 d^2}} = \boxed{\dfrac{1}{d}\sqrt{\dfrac{\lambda}{\pi^3 m_0}}}$

also ε_0

Problem 2 (30 points total):

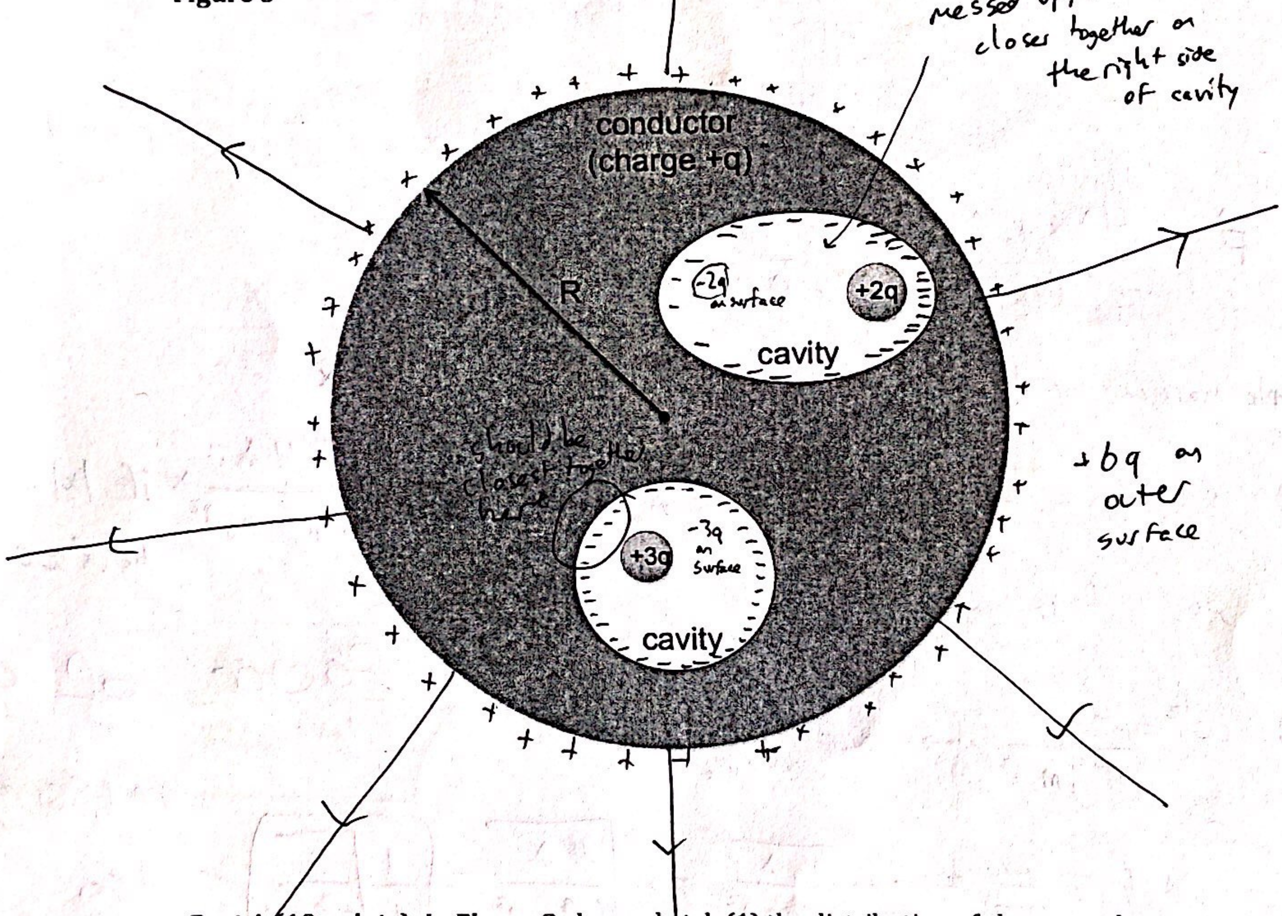
10/10

As shown in **Figure 3** below, a spherical conductor of radius R holds a net positive charge $+q$. The conductor also contains two internal cavities. One of the cavities contains a positive point charge $+2q$, and the other cavity contains a positive point charge $+3q$. For this problem, assume the space in the cavities and outside of the spherical conductor is vacuum.

$-2q + 3q + q_{outer} = q$

$q_{outer} = 6q$

Figure 3



Part A (10 points): In **Figure 3** above, sketch (1) the distribution of charge on the conductor and (2) the electric field lines outside of the conductor ($r > R$). Use "+" signs for positive charge and "-" signs for negative charge. The spacing between the signs should represent the relative charge density.

Part B (10 points): Determine the magnitude and direction of the electric field as a function of radial distance *r* from the conductor center (1) outside of the conductor $(r > R)$ and (2) inside of the material of the conductor. Express your answer in terms of the given parameters and fundamental constants.

10/10

(1) $E \cdot S.A. = \dfrac{Q_{encl}}{\varepsilon_0}$

$E \cdot 4\pi r^2 = \dfrac{q + 2q + 3q}{\varepsilon_0}$

$|\vec{E}| = \dfrac{6q}{4\pi r^2 \varepsilon_0} = \dfrac{3q}{2\pi r^2 \varepsilon_0}$ radially outward direction

$$\boxed{\vec{E} = E\hat{r} = \frac{3q}{2\pi r^2 \varepsilon_0}\hat{r}}$$

(2) el. field inside of the material of the conductors is 0.

$$\boxed{\vec{E} = \vec{0}}$$

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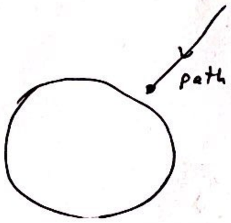
Part C (10 points): Taking the electrostatic potential to be zero infinitely far away, determine the potential as a function of radial distance *r* from the conductor center (1) outside of the conductor *(r > R)* and (2) inside of the material of the conductor. Express your answer in terms of the given parameters and fundamental constants.

(1) $V(r) = \int_{\infty}^{r} \vec{E}\cdot d\vec{r} = \int_{\infty}^{r} \frac{3q}{2\pi r^2 \varepsilon_0}(-1)\,dr = \frac{-3q}{2\pi\varepsilon_0}\int_{\infty}^{r} \frac{1}{r^2}\,dr$



$$= \frac{-3q}{2\pi\varepsilon_0}\left[\frac{-1}{r}\right]_{\infty}^{r} = \frac{3q}{2\pi\varepsilon_0}\left[\frac{1}{r} - \frac{1}{\infty}\right] = \frac{3q}{2\pi\varepsilon_0 r}$$

$$\boxed{V(r) = \frac{3q}{2\pi\varepsilon_0 r}}$$

(2) inside is same everywhere b.c. $\vec{E} = 0$

so $V_{inside} = V_{surface}$

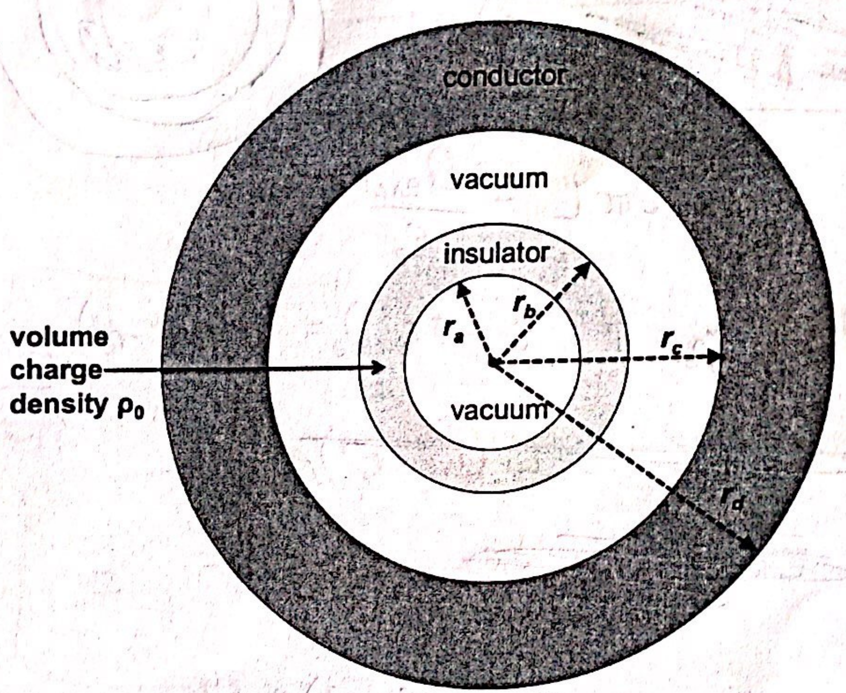
$$V_{surface} = V(R) = \frac{3q}{2\pi\varepsilon_0 R}$$

$$\boxed{\text{inside conductor: } V = \frac{3q}{2\pi\varepsilon_0 R}}$$

Problem 3 (30 points total):

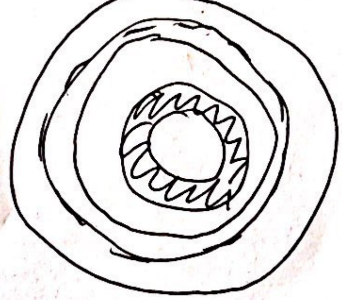
Figure 4 shows the cross-section of an infinitely long insulating cylindrical shell of inner radius r_a and outer radius r_b that has a uniform positive volume charge density ρ_0. The insulating shell is centered within an infinitely long uncharged conducting cylindrical shell of inner radius r_c and outer radius r_d.

Figure 4



Part A (10 points): Determine the surface charge density on (1) the inner surface of the conducting shell and (2) the outer surface of the conducting shell. Express your answers in terms of the given parameters and fundamental constants.

(1) ~~on inner surface: no charge b/c no charge in cavity~~



cylinder

$$\vec{E}\cdot 2\pi r L = \frac{Q_{encl}}{\varepsilon_0}$$

$$Q_{encl} = vol_{encl}\cdot \rho_0 = (\pi r_b^2 L - \pi r_a^2 L)\rho_0 = \pi L(r_b^2 - r_a^2)\,\rho_0$$

$$\cancel{\varepsilon_0 \vec{E}\cdot 2\pi r L = \pi L(r_b^2 - r_a^2)\rho_0}$$

$$Q_{inner} = -Q_{encl} = \pi L(r_b^2 - r_a^2)\rho_0 \quad +2$$

$$\frac{Q_{inner}}{\text{length}} = \text{chg density}_{inner} =$$

+1

$$\frac{Q_{encl}}{S.A.} = \frac{\cancel{\pi}\rho_0 \cancel{L}(r_b^2 - r_a^2)}{2\cancel{\pi} r_c \cancel{L}} = \boxed{\frac{-\rho_0(r_b^2 - r_a^2)}{2 r_c}} \quad (1)$$

$$(2)\ \frac{Q_{encl}}{S.A.} = \frac{-\cancel{\pi}\rho_0 \cancel{L}(r_b^2 - r_a^2)}{2\cancel{\pi} r_d \cancel{L}} = \boxed{\frac{\rho_0(r_a^2 - r_b^2)}{2 r_d}\ +2} \quad (2)$$

+2

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Part B (10 points): Determine the magnitude and direction of the electric field as a function of radial distance r from the center (1) inside the insulating shell ($r_a < r < r_b$) and (2) in the vacuum region between the insulating shell and the conducting shell ($r_b < r < r_c$). Express your answers in terms of the given parameters and fundamental constants.

(1) $\vec{E} \cdot 2\pi r L = \frac{Q_{encl}}{\varepsilon_0}$

$\oint \vec{E}\cdot\hat{n}\,dA = \vec{E} \cdot 2\pi r \cancel{L} = \frac{\rho_0 (\pi r^2 \cancel{L} - \pi r_a^2 \cancel{L})}{\varepsilon_0}$ +2

(1) $\boxed{\vec{E} = \frac{\rho_0 (r^2 - r_a^2)}{2 r \varepsilon_0}\hat{r}}$ +1

(2) $\vec{E} \cdot 2\pi r L = \rho_0 \ldots$ +1

$\boxed{\vec{E} = \rho_0 \frac{(r_b^2 - r_a^2)}{2 r \varepsilon_0}\hat{r}}$ +~~3~~4

8/10

Part C (10 points): Calculate the voltage (i.e., the magnitude of the potential difference) between the outer surface of the insulating shell ($r = r_b$) and the inner surface of the conducting shell ($r = r_c$). Express your answer in terms of the given parameters and fundamental constants.

$$V_{bc} = \int_b^c \vec{E} \cdot d\vec{r} \quad \times 2$$

$$= \frac{\rho_0 (r_b^2 - r_a^2)}{2\varepsilon_0} \int_b^c \frac{1}{r} dr = \boxed{\frac{\rho_0 (r_b^2 - r_a^2)}{2\varepsilon_0}\left[\ln\left(\frac{r_c}{r_b}\right)\right]}$$

$\times 2$ $\times 4$

$\frac{10}{10}$

SCORING

Problem 1:

Part A: 10 / 10
Part B: 10 / 10
Part C: 10 / 10
Part D: 8 / 10
Total: 38 / 40

Problem 2:

Part A: ____ / 10
Part B: ____ / 10
Part C: ____ / 10
Total: 30 / 30

Problem 3:

Part A: 7 / 10
Part B: 8 / 10
Part C: 10 / 10
Total: 25 / 30

Total Midterm #2 Score 93 / 100