Name: ______________________________

Student ID number: ______________________________

Oral interview time (see below): ______________________________

Discussion section: ______________

By signing below, you affirm that you have neither given nor received unauthorized help on this exam.

Signature: ______________________________

MIDTERM 2
PHYSICS 1B, SPRING 2020

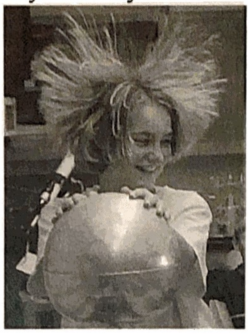
MAY 22, 2020

READ THE FOLLOWING CAREFULLY:

- ▷ Open book. Calculators will not be needed. Make sure that you work independently on the exam, i.e., no discussion with others.
- ▷ Oral interview time: choose a time period of 20 minutes between 11:59AM and 11:59PM on May 23 (Pacific Time) for possible oral interview (via the zoom office hour link). A random sample of students will be selected and be notified via email by 11:59AM on May 23 (Pacific Time). If all of the work on your submission is your own, you will have nothing to worry about.
- ▷ For non-response in the field of oral interview time, I may assess a score penalty.
- ▷ Make sure to submit your exam packet via GradeScope by 9AM May 23, Pacific Time. **No credit will be given for late submissions**.
- ▷ **You must make sure that your submission has exactly the same number of pages as the posted exam PDF (including this page)**.
- ▷ **You must justify your answers to each question**. Simply giving the correct answer without proper justification (can be brief) will not result in full credit.
- ▷ Mistakes in grading: If you find a mistake in the grading of your exam, alert the instructor within one week of the exams being returned.

[1.] **Short answer conceptual questions**. Provide *concise* answers to the questions below; you should write enough to explain your answer, but an essay is not required (most can be answered in 2-3 sentences).

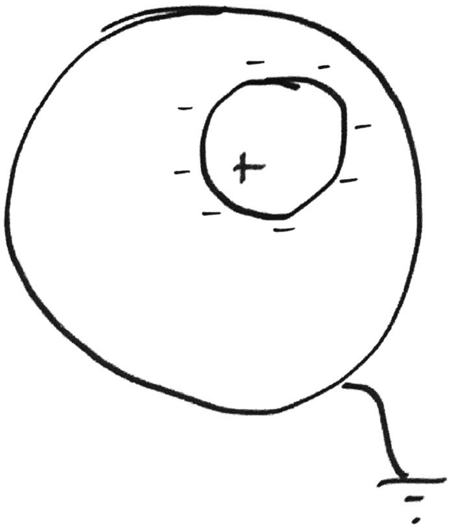
(a) (**10 pts**) During one of the in-class video demos, a kid stood on an insulator and placed her hands on the metal sphere of the Van der Graaf generator. This resulted in her hair standing on end (much like the below photo). Explain why this happened. If the kid wasn't standing on the insulator, would the same thing have happened? Why or why not?



The Van der Graaf generator generates a huge amount of positive charge. When the kid touches the generator, she will be positively charged. Since both the generator and the kid contain positive charges, they will repel each other. So the kid's ~~body~~ hair will stand on end.

If the kid wasn't standing on the insulator, excessive positive charges will be able to leave the kid's body through the conductor she is standing on. So the kid will not be positively charged. So the same thing will not happen.

(b) (**10 pts**) A solid conducting sphere has an internal spherical cavity which is offset from the center of the sphere. Inside the cavity, but offset from the center of the cavity, I place a positive point charge $+q$. The conducting sphere is connected to ground via a thin wire as shown. Describe (either by drawing or words) any charges that exist on the surfaces of the conductor; the interior surface around the cavity and the external surface. What is the total charge on the conducting sphere?

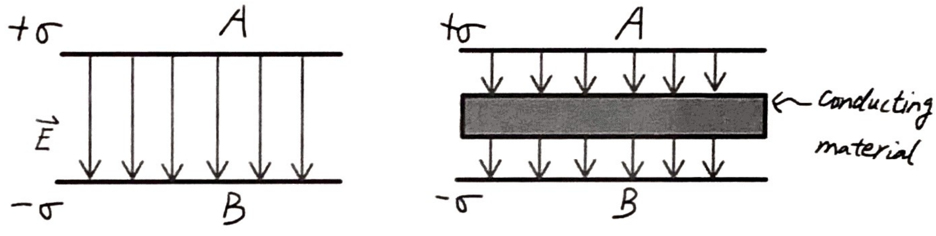


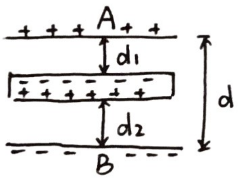
There will be a charge of $-q$ on the inner surface.

There will be no charge on the outer surface.

The total charge on the conducting sphere is $-q$.

(c) (**10 pts**) I have a capacitor consisting of two large parallel plates A and B, one charged positively (surface charge density $+\sigma$) and the other charged negatively (surface charge density $-\sigma$). If I insert a slab of conducting material as shown below, how does the absolute value of the potential difference between the two parallel plates A and B change? How does the capacitance change?





$$|\Delta V_{before}| = \left| \int_A^B \vec{E} \cdot d\vec{s} \right| = \frac{\sigma}{\varepsilon_0} \cdot d$$

After the conductor is inserted, the electric field will cause a charge density $-\sigma$ on its upper surface and $+\sigma$ on lower surface

So $$|\Delta V_{after}| = \left| \int_B^{lower} \vec{E} \cdot d\vec{s} + \int_{lower}^{upper} \vec{E} \cdot d\vec{s} + \int_{upper}^{A} \vec{E} \cdot d\vec{s} \right|$$

Because the electric field inside a conductor is zero,

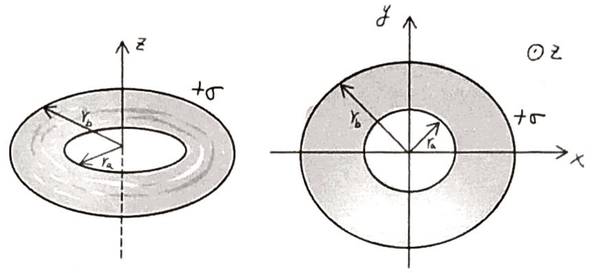
$$|\Delta V_{after}| = \frac{\sigma}{\varepsilon_0} d_1 + \frac{\sigma}{\varepsilon_0} d_2 + 0 < \frac{\sigma}{\varepsilon_0} d$$

So the absolute value of potential difference between A and B will decrease.

$\because$ $C = \frac{Q}{|\Delta V|}$, $|\Delta V|$ decreases and $Q = \sigma A$ remains the same

$\therefore$ The capacitance increases.

[2.] (**30 pts**) An annular ring with a uniform surface charge density $+\sigma$ sits in the xy plane, with its center at the origin of the coordinate. The annulus has an inner radius r_a and an outer radius r_b as shown below.



Side view   Top view

(a) What is the electric potential along the z-axis? Define the electric potential to be zero at infinity.

$$dq = \int_0^{2\pi} \sigma \cdot r dr d\theta = \sigma 2\pi r dr$$

$$dV = \frac{dq}{4\pi\varepsilon_0 R} = \frac{\sigma 2\pi r dr}{4\pi\varepsilon_0 \sqrt{r^2+z^2}}$$

$$V(z) = \frac{\sigma}{2\varepsilon_0}\int_{r_a}^{r_b} \frac{r dr}{\sqrt{r^2+z^2}} = \frac{\sigma}{2\varepsilon_0}\left(\sqrt{r_b^2+z^2} - \sqrt{r_a^2+z^2}\right)$$

(b) From part (a), what is the electric potential in the limit $|z| \gg r_a$, $|z| \gg r_b$? What is the electric potential in the limit $|z| \ll r_a$, $|z| \ll r_b$? Hint: use the binomial expansion $(1+x)^\alpha \approx 1 + \alpha x$ for $|x| \ll 1$.

When $|z| \gg r_a$, $|z| \gg r_b$:

$$V = \frac{\sigma}{2\varepsilon_0}\left[(r_b^2+z^2)^{\frac{1}{2}} - (r_a^2+z^2)^{\frac{1}{2}}\right]$$

$$= \frac{\sigma}{2\varepsilon_0}\cdot|z|\left[\left(\frac{r_b^2}{z^2}+1\right)^{\frac{1}{2}} - \left(\frac{r_a^2}{z^2}+1\right)^{\frac{1}{2}}\right]$$

$$\approx \frac{\sigma}{2\varepsilon_0}\cdot|z|\left(1+\frac{r_b^2}{2z^2} - 1 - \frac{r_a^2}{2z^2}\right)$$

$$= \frac{\sigma}{4\varepsilon_0|z|}(r_b^2 - r_a^2)$$

When $|z| \ll r_a$, $|z| \ll r_b$:

$$V = \frac{\sigma}{2\varepsilon_0}\left[r_b\left(1+\frac{z^2}{r_b^2}\right)^{\frac{1}{2}} - r_a\left(1+\frac{z^2}{r_a^2}\right)^{\frac{1}{2}}\right]$$

$$\approx \frac{\sigma}{2\varepsilon_0}\left[r_b\left(1+\frac{z^2}{2r_b^2}\right) - r_a\left(1+\frac{z^2}{2r_a^2}\right)\right]$$

(c) A particle of mass m and negative charge $-q$ is constrained to move along the z-axis. If I kick the particle with a small displacement about the point $z = 0$, the particle will execute simple harmonic motion. Find the particle's frequency of oscillation. Hint: use the electric potential in the limit $|z| \ll r_a$, $|z| \ll r_b$ from part (b) and $E_z = -\partial V/\partial z$, where V is the electric potential.

$$E_z = -\frac{\partial}{\partial z}\left(\frac{\sigma}{2\varepsilon_0}\left(r_b - r_a + \frac{z^2}{2r_b} - \frac{z^2}{2r_a}\right)\right)$$

$$= \frac{\sigma}{2\varepsilon_0}\left(\frac{z}{r_a} - \frac{z}{r_b}\right)$$

$$F = E_z(-q) = -\frac{\sigma q z}{2\varepsilon_0}\left(\frac{1}{r_a} - \frac{1}{r_b}\right) = m z''$$

$$z'' + \frac{\sigma q}{2\varepsilon_0 m}\left(\frac{1}{r_a} - \frac{1}{r_b}\right) z = 0$$

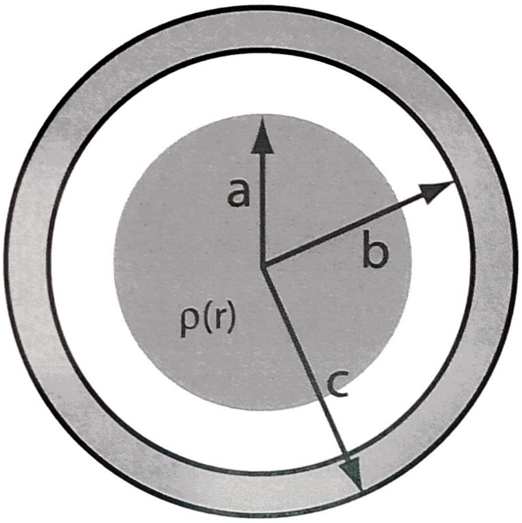
$$\omega = \sqrt{\frac{\sigma q}{2\varepsilon_0 m}\left(\frac{1}{r_a} - \frac{1}{r_b}\right)}$$

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{\sigma q}{2\varepsilon_0 m}\left(\frac{1}{r_a} - \frac{1}{r_b}\right)}$$

[3.] (**40 pts**) An insulating sphere of radius a has an embedded nonuniform charge density:

$$\rho(r) = \frac{\rho_o}{2}\left(1 + \frac{r}{a}\right)$$

where ρ_o is a positive constant (this charge density is only valid for $r \leq a$). Surrounding the sphere, and concentric with it, is a conducting shell of inner radius $b > a$ and outer radius c which is charged (not neutral). I do not know what the charge on the conductor is, but I am told that a measurement of the radial electric field right on the outer surface of the conducting shell (at $r = c + \epsilon$ where we take the limit $\epsilon \to 0$), shows that it is positive and equal to E_o.



(a) What is the total charge contained in the insulating sphere of radius a?

$$dQ = \rho(r)\, dV$$

$$Q = \int \rho(r) \cdot 4\pi r^2 dr$$

$$= \frac{\rho_o}{2} \cdot 4\pi \int_0^a r^2 + \frac{r^3}{a}\, dr$$

$$= \frac{7}{6}\pi a^3 \rho_o$$

(b) Find the electric field everywhere in space: for $r < a$, $a < r < b$, $b < r < c$ and $r > c$. Plot the electric field versus radius.

$\because$ The sphere is symmetric

$$\therefore\ E = \frac{\Phi_E}{A} = \frac{Q_{in}}{\varepsilon_0 A}$$

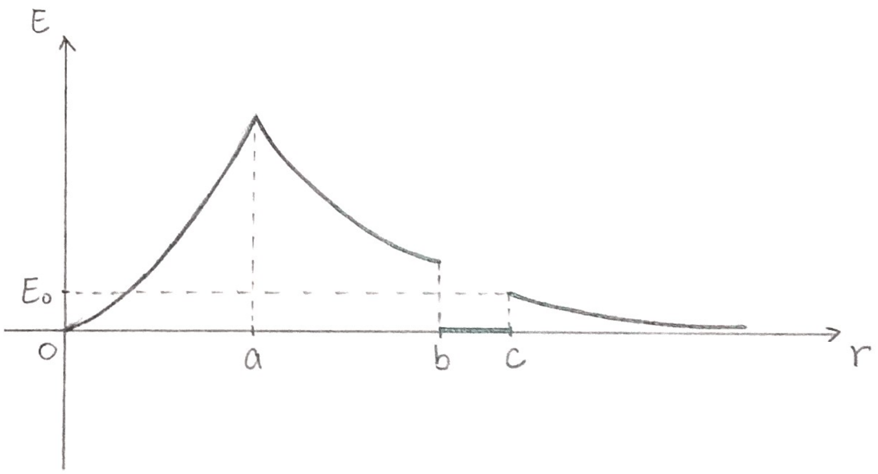
$$r < a:\quad E = \frac{2\pi\rho_0\left(\frac{r^3}{3}+\frac{r^4}{4a}\right)}{\varepsilon_0 \cdot 4\pi r^2} = \frac{\rho_0 r}{6\varepsilon_0} + \frac{\rho_0 r^2}{8a\varepsilon_0}$$

$$a < r < b:\quad E = \frac{\frac{7}{6}\pi a^3 \rho_0}{\varepsilon_0 \cdot 4\pi r^2} = \frac{7\rho_0 a^3}{24\varepsilon_0 r^2}$$

$$b < r < c:\quad E = 0$$

$$r > c:\quad E_0 = \frac{Q_{in}}{4\pi c^2 \varepsilon_0} \qquad Q_{in} = E_0 \cdot 4\pi c^2 \varepsilon_0$$

$$E = \frac{E_0 \cdot 4\pi c^2 \varepsilon_0}{4\pi r^2 \varepsilon_0} = \frac{c^2}{r^2} E_0$$



(c) What is the total charge on the conducting shell in terms of the given information? What is the charge density on the inner and outer surfaces?

$$Q_{bc} = Q_{tot} - Q_a$$

$$= E_0 \cdot 4\pi c^2 \varepsilon_0 - \frac{7}{6}\pi a^3 \rho_0$$

$$Q_{inner} = -Q_a = -\frac{7}{6}\pi a^3 \rho_0$$

$$Q_{outer} = Q_{bc} - Q_{inner}$$

$$= E_0 \cdot 4\pi c^2 \varepsilon_0$$

$$\sigma_{inner} = \frac{-\frac{7}{6}\pi a^3 \rho_0}{4\pi b^2} = -\frac{7a^3 \rho_0}{24 b^2}$$

$$\sigma_{outer} = \frac{E_0 \cdot 4\pi c^2 \varepsilon_0}{4\pi c^2} = E_0 \varepsilon_0$$

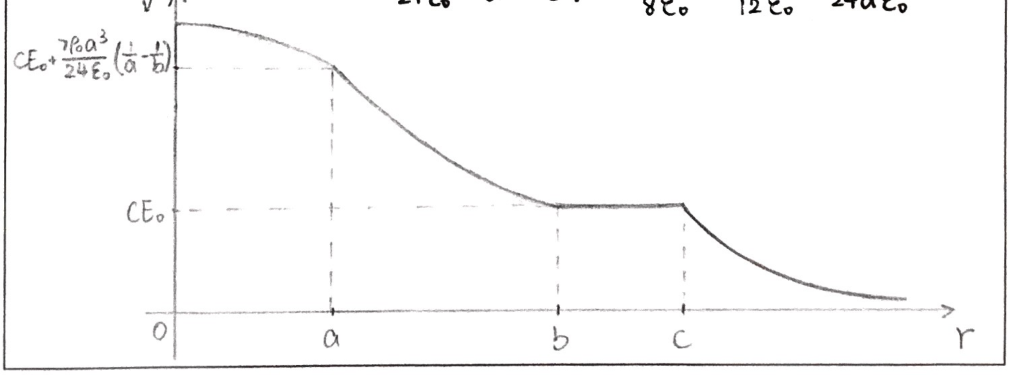
(d) Find the electric potential everywhere in space: for $r < a$, $a < r < b$, $b < r < c$ and $r > c$. Plot the electric potential versus radius. Define the electric potential to be zero at infinity.

$r > c$:
$$V_\infty - V_r = \int_\infty^r \vec{E}\cdot d\vec{s} = \int_\infty^r \frac{c^2}{r^2} E_0\, dr = -\frac{c^2}{r} E_0$$
$$V_r = \frac{c^2}{r} E_0$$

$b < r < c$:
$$V_\infty - V_r = \int_\infty^c \vec{E}\cdot d\vec{s} + \int_c^r \vec{E}\cdot d\vec{s}$$
$$= -cE_0 + 0$$
$$V_r = cE_0$$

$a < r < b$:
$$V_\infty - V_r = \int_\infty^c \vec{E}\cdot d\vec{s} + \int_c^b \vec{E}\cdot d\vec{s} + \int_b^r \vec{E}\cdot d\vec{s}$$
$$= -cE_0 + 0 + \int_b^r \frac{7\rho_0 a^3}{24\varepsilon_0 r^2}\, dr$$
$$= -cE_0 - \frac{7\rho_0 a^3}{24\varepsilon_0}\left(\frac{1}{r} - \frac{1}{b}\right)$$
$$V_r = cE_0 + \frac{7\rho_0 a^3}{24\varepsilon_0}\left(\frac{1}{r} - \frac{1}{b}\right)$$

$r < a$:
$$V_\infty - V_r = \int_\infty^c \vec{E}\cdot d\vec{s} + \int_c^b \vec{E}\cdot d\vec{s} + \int_b^a \vec{E}\cdot d\vec{s} + \int_a^r \vec{E}\cdot d\vec{s}$$
$$= -cE_0 - \frac{7\rho_0 a^3}{24\varepsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right) + \int_a^r \frac{\rho_0 r}{6\varepsilon_0} + \frac{\rho_0 r^2}{8a\varepsilon_0}\, dr$$
$$= -cE_0 - \frac{7\rho_0 a^3}{24\varepsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right) + \frac{\rho_0 r^2}{12\varepsilon_0} + \frac{\rho_0 r^3}{24a\varepsilon_0} - \frac{\rho_0 a^2}{8\varepsilon_0}$$
$$V_r = cE_0 + \frac{7\rho_0 a^3}{24\varepsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right) + \frac{\rho_0 a^2}{8\varepsilon_0} - \frac{\rho_0 r^2}{12\varepsilon_0} - \frac{\rho_0 r^3}{24a\varepsilon_0}$$



(e) What is the work that I do to assemble charges in this configuration, i.e., the total electric potential energy of this configuration? Use your answer to part (d), and the relationship between the electric potential energy and electric potential for a collection of charges.

$$W = \frac{1}{2}\int V\,dq$$

$$= \frac{1}{2}\left(Q_c V_c + Q_b V_b + \int_0^a V\,dq\right)$$

$$= \frac{1}{2} cE_0\left(E_0 \cdot 4\pi c^2 \varepsilon_0 - \frac{7}{6}\pi a^3 \rho_0\right)$$

$$+ \frac{1}{2}\int_0^a \left[cE_0 + \frac{7\rho_0 a^3}{24\varepsilon_0}\left(\frac{1}{a}-\frac{1}{b}\right) + \frac{\rho_0 a^2}{8\varepsilon_0} - \frac{\rho_0 r^2}{12\varepsilon_0} - \frac{\rho_0 r^3}{24 a\varepsilon_0}\right]$$

$$\cdot\, 4\pi r^2 \rho(r)\,dr$$

$$= \frac{1}{2} cE_0\left(E_0 \cdot 4\pi c^2 \varepsilon_0 - \frac{7}{6}\pi a^3 \rho_0\right) +$$

$$\frac{1}{2}\left(cE_0 + \frac{7\rho_0 a^3}{24\varepsilon_0}\left(\frac{1}{a}-\frac{1}{b}\right) + \frac{\rho_0 a^2}{8\varepsilon_0}\right)\frac{7}{6}\pi a^3 \rho_0 - \frac{73\rho_0^2 a^5}{1680\varepsilon_0}$$

$$= 2\pi c^3 \varepsilon_0 E_0^2 + \frac{49\pi\rho_0^2}{288\varepsilon_0}\left(a^5 - \frac{a^6}{b}\right) + \frac{7\pi\rho_0^2}{96\varepsilon_0}a^5 - \frac{73\rho_0^2 a^5}{1680\varepsilon_0}$$

$$= 2\pi c^3 \varepsilon_0 E_0^2 + \frac{503\pi\rho_0^2 a^5}{2520\varepsilon_0} - \frac{49\pi\rho_0^2 a^6}{288\varepsilon_0 b}$$

(f) What is energy stored in the electric field? Use your answer to part (b), and the relationship between energy density and electric field.

Compare the electric field energy in this part with the total electric potential energy in part (e). Are they the same or not?

$$U = \int \frac{\varepsilon_0 E^2}{2} \, d\forall$$

$$U_{oa} = \int_0^a \frac{\varepsilon_0}{2} \left(\frac{\rho_0 r}{6\varepsilon_0} + \frac{\rho_0 r^2}{8a\varepsilon_0} \right)^2 \cdot 4\pi r^2 \, dr$$

$$= \frac{33\pi \rho_0^2}{1120\,\varepsilon_0} a^5$$

$$U_{ab} = \int_a^b \frac{\varepsilon_0}{2} \left(\frac{7\rho_0 a^3}{24\varepsilon_0 r^2} \right)^2 \cdot 4\pi r^2 \, dr = \frac{49 \rho_0^2 \pi}{288\,\varepsilon_0} \left(a^5 - \frac{a^6}{b} \right)$$

$$U_{bc} = 0$$

$$U_{c\infty} = \int_c^\infty \frac{\varepsilon_0}{2} \left(\frac{c^2}{r^2} E_0 \right)^2 \cdot 4\pi r^2 \, dr = 2\pi c^3 \varepsilon_0 E_0^2$$

$$U_{tot} = 2\pi c^3 \varepsilon_0 E_0^2 + \frac{503}{2520\varepsilon_0} \pi \rho_0^2 a^5 - \frac{49\pi \rho_0^2 a^6}{288\,\varepsilon_0 b}$$

The answers of (e) and (f) are the same.