Physics 1A Midterm 1

Vectors; 1D, 2D, and 3D Kinematics

April 21, 2020

Question 3: Interpreting Plots of 1D Kinematic Quantities [15 pts]

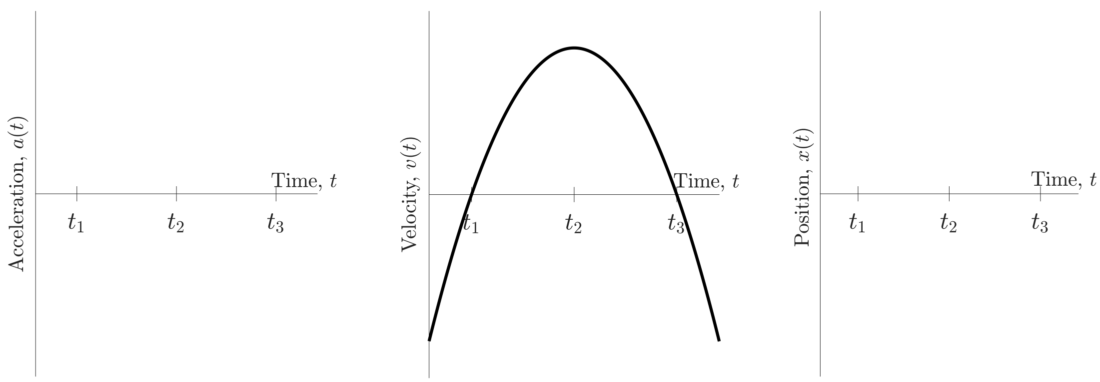

Figure 1: Graphs of $x(t)$, $v(t)$ and $a(t)$.

In the velocity vs. time plot above, the velocity is a quadratic function of time; i.e. $v(t) = at^2 + bt + v_0$, for some (unspecified—this plot is just qualitative) constants a, b, and v_0.

1. Given this $v(t)$, reproduce the blank coordinate systems provided in Fig. 1 and draw in the acceleration vs. time curve $a(t)$ and the position vs. time curve $x(t)$. You may or may not have to make arbitrary choices in your drawing, but make sure that your plots are qualitatively consistent with all features of the $v(t)$ plot, paying particular attention to what is happening at t_1, t_2, and t_3. [8 pts.]

2. (i) In which time intervals is the object slowing down (decreasing in speed)? (ii) In which time intervals is it speeding up (increasing in speed)? (iii) At which time points, if any, is the acceleration zero? (iv) At which time points, if any, does the position in x reach a local minimum? (Recall that a quantity increases in value on both sides of its local minima.) (v) Does the object ever return to its position at t_3? [7 pts.]

Question 4: Vector Operations [10 pts.]

1. Draw your own arbitrary vectors $\vec{A}$ and $\vec{B}$ in an x-y (Cartesian) coordinate plane. Do not make them parallel, anti-parallel, or perpendicular to each other. Using graphical methods of vector algebra, show how how you would construct the resultant vector $\vec{R} = 2\vec{A} - \vec{B}$ from its summand vectors. (Your drawing need not be perfect! You are expected to capture the basic procedures of multiplying a vector by a scalar and subtracting another vector, not to get the resulting lengths and directions exactly right.) [4 pts.]

2. Write your vectors in the form $\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath}$ and $\vec{B} = B_x\hat{\imath} + B_y\hat{\jmath}$, choosing numerical values for the vector components. The exact values can be whatever you want - they're your vectors! - but they should be qualitatively consistent with the way you drew the vectors on the plane. Now compute vector $\vec{R} = 2\vec{A} - \vec{B}$ and give the result in polar coordinates, providing the magnitude R and direction θ (measured counterclockwise with respect to the positive x-axis) of the resultant vector. [3 pts.]

3. Compute $c = \vec{A} \cdot \vec{B}$. Find the angle between $\vec{A}$ and $\vec{B}$. [3 pts.]

Question 5: Kinematics Trolley Problem [25 pts.]

A dastardly physics professor has deposited you in a diabolical dilemma. You are standing next to a forked track. An out-of-control trolley car is barrelling down the line with initial speed v_0. Within your reach is a lever whose position determines whether the trolley takes Track A or B. One of your TAs is chained to Track A, and the other is chained to Track B. Both TAs are located the same distance d from the fork. The evil but truthful professor tells you that on only one of the tracks will the train reach a stop before it runs over the TA.

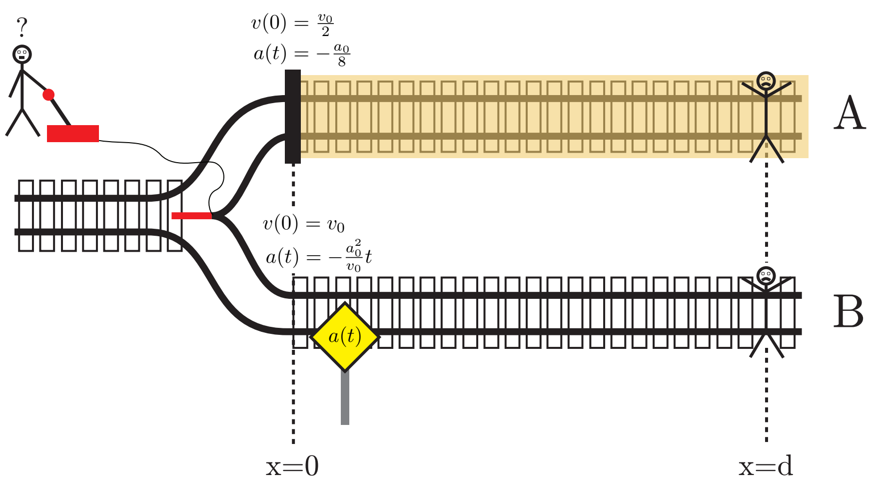

Figure 2: The trolley problem.

The rules for the tracks are as follow:

1. At the beginning of Track A is a speed bump that reduces the trolley's initial speed from v_0 to $v_0/2$. The rest of the track is covered in sand that causes the trolley to slow down with a constant acceleration of $-a_0/8$ until it reaches a stop ($v_A = 0$).

2. Next to Track B is a sign indicating that the trolley's time-dependent acceleration on the track is $a_B(t) = -\frac{a_0^2}{v_0}t$ until the trolley reaches a stop ($v_B = 0$).

3. Once the trolley reaches a stop on either track, it stays stopped forever ($a_{A,B} = 0$ if $v_{A,B} = 0$).

Problems:

1. Find the stopping positions $x_{f,A}$ and $x_{f,B}$ reached by the trolley on tracks A and B, respectively, in terms of the constants provided (v_0, a_0, and d: you may or may not need all of these). Given that on one of the tracks, the train will stop before running over the imperilled TA, in which position, A or B, should you place the lever to save both TAs? [12 pts.]

2. After saving the TAs, you try to free them, only to find the chains secured with an alphanumeric combination lock. The TAs tell you that they overheard the professor muttering that the combination for the two locks is given by $v_{\text{avg},A}$ and $v_{\text{avg},B}$, the average velocity of the train on tracks A and B, respectively, between the time when the train enters that fork of the track and the time when the it arrives at a complete stop. Find $v_{\text{avg},A}$ and $v_{\text{avg},B}$, expressed in terms of v_0, a_0, and d (you may or may not need all of these constants). Explain in words why it is or is not possible for this result to be consistent with your result from part (a). [7 pts.]

3. As you are walking away, the villainous professor appears again. "I will release you from this hypothetical scenario," she says, "*if* you can compute the instantaneous velocity of the trolley on each track at a time halfway to the stopping time," i.e., $v_A(T_A/2)$ and $v_B(T_B/2)$, where T_A is the time taken by the trolley to arrive at a complete stop on track A and T_B is the time taken by the trolley to arrive at a complete stop on Track B. Your answer should be expressed in terms of v_0, a_0, and d (you may or may not need all of these constants). [6 pts.]

Question 6: Shooting Hoops [25 pts.]

Adam and Beth are playing basketball and trying to make half-court shots (a long shot, from the halfway line) with the ball going straight into the basket without touching the backboard. Adam says that in order to minimize the effort it takes to make a basket, as you move farther away from the basket, you should aim your shots at a lower angle than if you were closer to the basket. Beth says that as you move farther away, it's better to aim higher than when you were close. Let's find out who is right.

Consider the half-court shot. The ball is released from a height h, a horizontal distance L away from the basket. The initial speed of the ball is v_0 at an angle θ relative to horizontal. The basket is at a height d. Ignore the effect of air resistance in this problem.

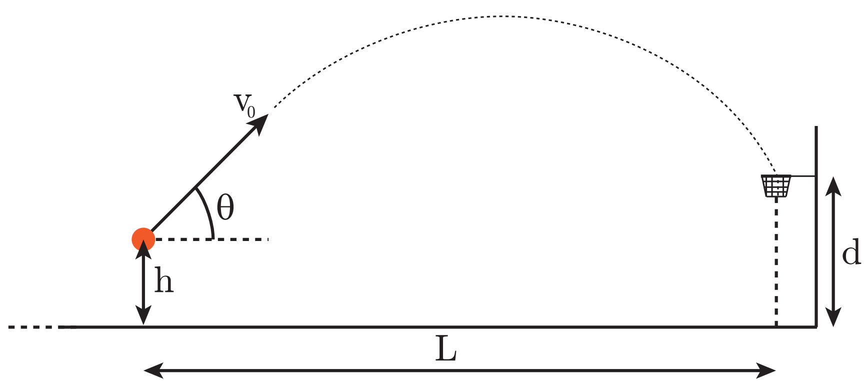

Figure 3: Hoops problem.

1. Write down the equations of motion $x(t)$ and $y(t)$ for the ball. Find the trajectory $y(x)$ by eliminating the variable t. [4 pts.]

2. By stipulating that the ball goes through the hoop, show that the equation for the required initial speed in terms of the initial angle is [4 pts.]

$$v_0 = \sqrt{\frac{gL^2}{2[L\sin\theta\cos\theta + (h-d)\cos^2\theta]}}. \tag{1}$$

3. We want to find the trajectory with the least effort. Given that we want to **minimize** v_0, inspect Eq. 1 to identify the simplest expression, in terms of the initial angle θ, which we should **maximize.** [2 pts.]

4. Maximize the previous expression to show that the expression for the optimum angle θ is [4 pts.]

$$\tan(2\theta) = \frac{L}{h-d}. \tag{2}$$

5. Because the tangent function is periodic, we must be careful to choose the correct range when taking its inverse, which sometimes means that we need to add $\pi\,\mathrm{rad} = 180°$ to the angle returned by a calculator's $\tan^{-1}$ function. By inspecting Fig. 3, determine what range of angles 2θ makes physical sense as the argument of the tangent function in Eq. 2, and explain your reasoning in a sentence or two. [2 pts.]

6. Assuming that $h < d$, find the angle θ in the following limiting cases: $L \to 0$ and $L \to \infty$. Do your answers make physical sense? Explain in words why or why not. By examining these results, determine who is correct: Adam or Beth? Justify your answer in a sentence or two. [7 pts.]

7. Given that a half court shot has $L = 14.3$ m, a basketball rim is a distance $d = 3.05$ m above the ground and the ball is released from a height $h = 2.0$ m, find the angle θ for least effort. Assuming $g = 9.8$ m/s^2, at what speed v_0 should you throw the ball for this angle? [2 pts.]

Question 7: Circular Carnival Ride [25 pts.]

A Gravitron is a cylindrical carnival ride in which people stand against a circular outer wall of radius R while the ride spins up to a constant speed about its axis. After the ride reaches its maximum speed, the floor slowly tilts upward while the ride's rotation provides enough centripetal acceleration a_c to keep the riders pressed against the wall. Ride restraints and friction ensure that the riders remain in a fixed position along the circumference of the ride, rather than slipping sideways.

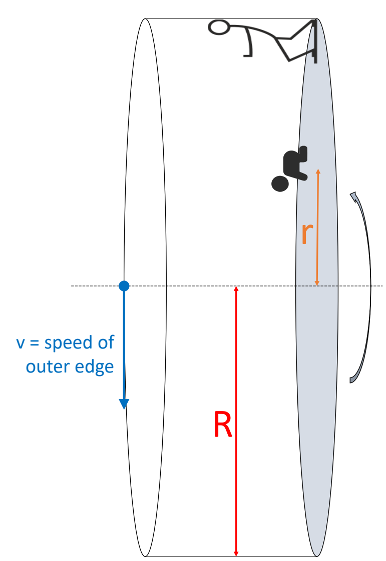

Figure 4: Diagram of Gravitron in its vertical operating position.

1. Assume that the ride tilts upward until the floor is completely vertical, and suppose that you are a Gravitron rider at the top of your circular trajectory. What condition must the ride's centripetal acceleration a_c meet to ensure you stay against the wall rather than starting to fall downward at this point in your trajectory? Your answer should be expressed in the form $a_c >, <,$ or $=$ some quantity a_{const}, where a_{const} is given in terms of fundamental physical and numerical constants. Explain your answer in words. [5 pts.]

2. If the ride has radius R, what is the speed v_0 of the outer wall needed to meet the condition specified in part (a)? Is this speed a minimum or maximum? Express your answer in terms of R and fundamental physical and numerical constants. [3 pts.]

   If you were unable to complete part (a), express your answer in terms of R and a_{const}.

3. Suppose the ride is 10. m in diameter ($R = 5.0$ m). Compute the period T of the ride (the time taken to complete a full turn) assuming that the ride produces the centripetal acceleration found in part (a). [3 pts.]

   If you were unable to complete part (a), express your answer algebraically in terms of a_{const}, then specify a value of your choice for a_{const} and plug it in to obtain a numerical answer.

4. For added fun, Gravitrons typically provide about 3 times the acceleration found in part (a) for riders on the outer wall. Suppose you are riding a Gravitron that is operating under these conditions when a mischievous child slips out of the ride restraints and starts climbing the floor towards the center axle of the ride. Show that the child's centripetal acceleration decreases as she gets closer to the axle, making sure to justify your work in words. (Assume that the child is climbing much more slowly than the ride is spinning, so that she is always instantaneously in a state of uniform circular motion.) [7 pts.]

5. Still assuming that the ride is operating with a centripetal acceleration at the outer wall that is 3 times the value you found in part (a) (*Note that you may use a_{const} and the same value you chose for it in part (c) if you were unable to complete part (a)*) and that the ride is rotating in its vertical position, you must catch the child before she arrives at radius $r = \alpha R$ from the center to keep her from falling downwards when she is at the top of her trajectory. Find the numerical value of α. [7 pts.]

Physics 1A Midterm 1—Solutions

Vectors; 1D, 2D, and 3D Kinematics

April 21, 2020

Question 3: Interpreting Plots of 1D Kinematic Quantities [15 pts.]

1.

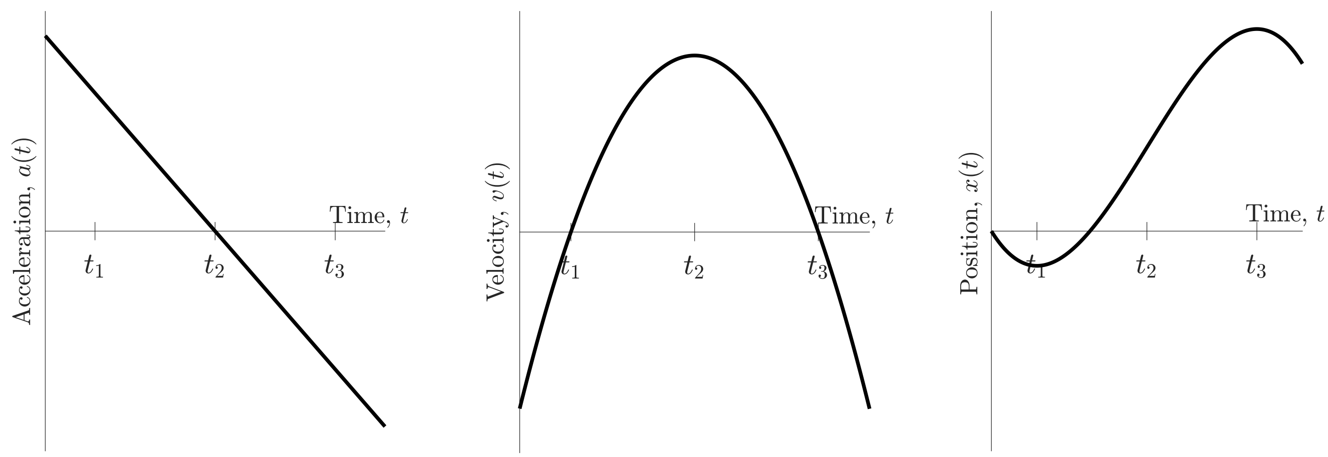

Figure 1: Plots of acceleration, velocity and position.

Left-hand plot, acceleration should have the following features:
Linear
Negative slope
Crossing at t_2

Right-hand plot, position should have the following features:
Turning points at t_1 and t_3
Signs of slopes
Linear at t_2

Total points = [+8]

2. (i) $0 \to t_1$, $t_2 \to t_3$
   (ii) $t_1 \to t_2$, $> t_3$
   (iii) t_2
   (iv) t_1
   (v) No

   Total points = [+7]

Question 4: Vector Operations [10 pts.]

1.

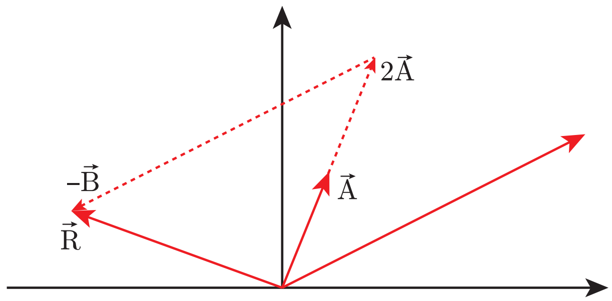

Figure 2: Example vectors.

A correct answer will:
Follow vector instructions
Show scalar multiplication with qualitative accuracy
Show subtraction with qualitative accuracy
Show resultant vector

Total points: [+4]

2. Example:

$$\vec{A} = 1\hat{i} + 2\hat{j}$$
$$\vec{B} = 4\hat{i} + 3\hat{j}$$

$$\vec{R} = (2-4)\hat{i} + (4-3)\hat{j}$$

$$|\vec{R}| = \sqrt{5} \qquad \theta = 150°$$

A correct answer will:
Have a reasonable component decomposition
Compute the value of $|\vec{R}|$
Compute the value of θ

Total points: [+3]

3. Example:

$$\begin{aligned} c &= \vec{A} \cdot \vec{B} \\ &= 4 + 6 \\ &= 10 = |A||B|\cos\theta \qquad \text{Formula for dot product} \end{aligned}$$

$$|A| = \sqrt{5}$$
$$|B| = 5$$

$$\theta = \cos^{-1}\left(\frac{10}{5\sqrt{5}}\right) = \cos^{-1}\left(\tfrac{2}{\sqrt{5}}\right) = 27°$$

Total points: [+3]

Question 5: Kinematics Trolley Problem [25 pts.]

1. Track A: Find the stopping distance by using the position-velocity relationship for constant acceleration and setting the final velocity equal to zero.

$$\begin{aligned}
v_{\mathrm{f,A}}^2 - \left(\tfrac{v_0}{2}\right)^2 &= 2a(x_{\mathrm{f,A}}) && \text{From kinematic equation (iii) for constant acc.} \\
-\left(\tfrac{v_0}{2}\right)^2 &= -2\tfrac{a_0}{8}x_{\mathrm{f,A}} && \text{Set } v_{\mathrm{f,A}} = 0 \\
\frac{v_0^2}{a_0} &= x_{\mathrm{f,A}} && \text{Solve for } x_{\mathrm{f,A}}
\end{aligned}$$

Track B: In the case of non-constant acceleration, use the general kinematic relations between acceleration, velocity, and position to find the displacement at the stopping time.

$$\begin{aligned}
v_{\mathrm{B}}(t) &= v_0 - \int_0^t \frac{a_0^2}{v_0}t' \ \mathrm{d}t' && \text{From general kinematic relation between velocity and acceleration} \\
&= v_0 - \frac{a_0^2}{2v_0}t^2 \\
v_{\mathrm{B}}(T_{\mathrm{B}}) = 0 &= \frac{a_0^2}{2v_0}T_{\mathrm{B}}^2 && \text{Find stopping time } T_B \text{ by letting } v_B(T_B) = 0 \\
\Rightarrow T_{\mathrm{B}} &= \sqrt{2}\frac{v_0}{a_0} \\
x_{\mathrm{B}}(t) &= \int_0^t v_{\mathrm{B}}(t') \ \mathrm{d}t' = \int_0^t v_0 - \frac{a_0^2}{2v_0}t'^2 \ \mathrm{d}t' && \text{From general kinematic relation between } v \text{ and } x \\
&= v_0 t - \frac{a_0^2}{6v_0}t^3 \\
x_{\mathrm{f,B}} = x(T_{\mathrm{B}}) &= v_0 T_{\mathrm{B}} - \frac{a_0^2}{6v_0}T_{\mathrm{B}}^3 && \text{Solve for } x_{\mathrm{f,B}} \text{ by substituting in the value of } T_B \\
&= v_0\left(\sqrt{2}\frac{v_0}{a_0}\right) - \frac{a_0^2}{6v_0}\left(\sqrt{2}\frac{v_0}{a_0}\right)^3 \\
&= \sqrt{2}\left(\frac{v_0^2}{a_0} - \frac{v_0^2}{3a_0}\right) \\
x_{\mathrm{f,B}} &= \frac{2\sqrt{2}}{3}\frac{v_0^2}{a_0}
\end{aligned}$$

By comparing $x_{\mathrm{f,B}}$ and $x_{\mathrm{f,A}}$, we see that both are proportional to v_0^2/a_0, so we only need to compare the constants of proportionality: $\frac{2\sqrt{2}}{3} < 1$, so we should set the lever to position B.

Total points: [+12]

2. The definition of average velocity is: $v_{\text{avg}} = \frac{\Delta x}{\Delta t}$.

   Track A: At constant acceleration, $v_{\text{avg}} = \frac{v_{\text{i}}+v_{\text{f}}}{2}$, so

   $$\begin{aligned} v_{\text{avg,A}} &= \frac{1}{2}\frac{v_0}{2} \\ &= \frac{v_0}{4} \end{aligned}$$

   Track B: In part 5.1 we found $T_{\text{B}} = \frac{\sqrt{2}v_0}{a_0}$, so

   $$v_{\text{avg,B}} = \frac{x_{\text{f,B}}}{T_{\text{B}}} = \frac{2\sqrt{2}v_0^2}{3a_0}\frac{a_0}{\sqrt{2}v_0} = \tfrac{2}{3}v_0$$

   This result is consistent with part 5.1. Although the average speed on track B was higher, the trolley also had a larger acceleration. This meant that the denominator Δt was much shorter for track B than track A, producing a shorter stopping distance $\Delta x = v_{\text{avg}}\Delta t$ in spite of the larger average speed.

   Total points: [+7]

3. Track A: We first need to find the stopping time T_A for this track, and then we can evaluate the velocity at this time.

   $$\begin{aligned} v(T_{\text{A}}) &= v(0) + aT_{\text{A}} && \text{From kinematic relation (i) for constant acceleration} \\ 0 &= \tfrac{v_0}{2} - \tfrac{a_0}{8}T_{\text{A}} && \text{Set } v(T_{\text{A}}) = 0 \\ \tfrac{v_0}{2} &= \tfrac{a_0}{8}T_{\text{A}} && \text{Solve for } T_{\text{A}} \\ T_{\text{A}} &= 4\frac{v_0}{a_0} \end{aligned}$$

   $$\begin{aligned} v_{\text{A}}(T_{\text{A}}/2) &= \tfrac{v_0}{2} - \tfrac{a_0}{8}\tfrac{T_{\text{A}}}{2} && \text{Plug } T_{\text{A}}/2 \text{ back into the expression for } v_{\text{A}}(t) \\ v_{\text{A}}(T_{\text{A}}/2) &= \frac{v_0}{4} \end{aligned}$$

   Track B: From 5.1, we know that $T_{\text{B}} = \frac{\sqrt{2}v_0}{a_0}$, and $v_{\text{B}}(t) = v_0 - \frac{a_0^2}{2v_0}t^2$. All we need to do now is substitute the expression for $T_{\text{B}}/2$ into the expression for $v_{\text{B}}(t)$.

   $v_{\text{B}}(T_{\text{B}}/2) = v_0 - v_0/4 = \frac{3}{4}v_0$

   Total points: [+6]

Question 6: Shooting Hoops [25 pts.]

1. The equations of motion for the x and y coordinates are:

   $$x(t) = v\cos(\theta)t$$
   $$y(t) = v\sin(\theta)t - \frac{1}{2}gt^2 + h.$$

   Hence, $t = x/v\cos\theta$ and the trajectory is

   $$\Rightarrow y = x\tan\theta - \frac{gx^2}{2v^2\cos^2\theta} + h.$$

   Total points: [+4]

2. Here we set $x = L$ and $y = d$ to give

   $$d = L\tan\theta - \frac{gL^2}{2v^2\cos^2\theta} + h.$$

   Rearranging this for v gives the desired expression:

   $$v = \sqrt{\frac{gL^2}{2L\sin\theta\cos\theta + 2(h-d)\cos^2\theta}}.$$

   Total points: [+4]

3. Given that we want to minimize v, we must maximize the denominator in the above expression, i.e. maximize

   $$f(\theta) \equiv L\sin\theta\cos\theta + (h-d)\cos^2\theta.$$

   Total points: [+2]

4. We can differentiate the above expression to give

   $$\frac{\partial f}{\partial \theta} = L(\cos^2\theta - \sin^2\theta) - (h-d)2\sin\theta\cos\theta.$$

   Using the double-angle formulae for sine and cosine, we have

   $$f(\theta) = L\cos 2\theta - (h-d)\sin 2\theta.$$

   Setting the derivative to zero and solving we have

   $$\theta = \frac{1}{2}\text{atan}\left(\frac{L}{h-d}\right) \Rightarrow \tan 2\theta = \frac{L}{h-d}$$

   Total points: [+4]

5. θ should be between $0°$ and $90°$ in order to be aimed towards the basket, so $0 < 2\theta \leq 180°(=\pi)$.
   Total points: [+2]

6. $L \to 0 \Rightarrow \tan(2\theta) \to 0 \Rightarrow 2\theta = 0°$ or $180°$. $0°$ does not make physical sense for the case where you are standing under the basket, so the answer must be $L \to 0 \Rightarrow \theta \to 90°$. This makes sense as it implies that you should shoot the ball nearly straight up as you approach the basket.

$L \to \infty \Rightarrow 2\theta = 90^\circ \Rightarrow \theta = 45^\circ$. This implies that in the limit that you are very far from the basket, you should aim your shot at 45°. This is consistent with the result for achieving the maximum range of a projectile fired at a given initial velocity.

These results show that Adam is correct: You should lower your aim as you move away from the basket, but never below 45° above the horizontal.

Total points: [+7]

7. Plugging in numbers we have

$$\begin{aligned}\theta &= \frac{1}{2}\left[\tan^{-1}\left(\frac{14.3 \text{ m}}{2 \text{ m} - 3.05 \text{ m}}\right) + \pi\right] \\ &= \frac{1}{2}\left[\tan^{-1}(-13.62) + \pi\right] = 0.82 \text{ radians} = 47^\circ.\end{aligned}$$

Using our equation for v_0 we have

$$v = \sqrt{\frac{(9.81 \text{ m/s}^2) \times (14.3 \text{ m})^2}{2 \times (14.3 \text{ m}) \times \sin(0.82)\cos(0.82) + 2(2.0 \text{ m} - 3.05 \text{ m})\cos^2(0.82)}} = 12 \text{ m/s}.$$

Total points: [+2]

Question 7: Circular Motion Carnival Ride [25 pts.]

1. Centripetal acceleration is the radial acceleration required to keep an object moving in a circular trajectory at a constant speed. At the top of the trajectory, the acceleration due to gravity is the minimum radial acceleration experienced by the rider. (It may be larger than this is the ride is spinning fast enough for $\frac{v^2}{r}$ to exceed g.) If the acceleration due to gravity exceeds $\frac{v^2}{r} = a_C$, then the rider will no longer remain in a circular trajectory of radius R, but will accelerate towards the center of the circle. The condition the ride must meet is therefore:

$$a_c \geq g$$

Total points: [+5]

2. By setting the formula for centripetal acceleration equal to the limit specified in part 7.1, $\frac{v_0^2}{R} = g \Rightarrow v_0 = \sqrt{gR}$. This is the **minimum** speed such that $a_C \geq g$.

Total points: [+3]

3. Use $a_C = \frac{4\pi^2 R}{T^2} \Rightarrow T = \sqrt{\frac{4\pi^2 R}{a_C}} = 4.5$ s.

Total points: [+3]

4. We know that $a_C = \frac{4\pi^2 r}{T^2}$. Because the child's rotational period stays the same as she crawls toward the center, T remains constant while r decreases, so a_C decreases as the child crawls toward the center.

Total points: [+7]

5. Acceleration at outside, $a_{C,\text{outside}} = 3g = \frac{v_0^2}{R}$.

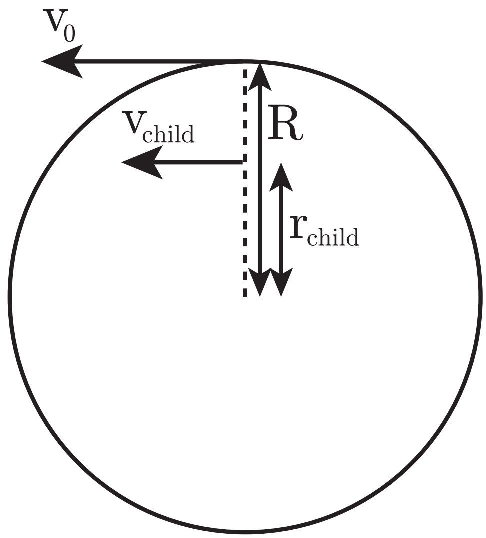

Figure 3: Sketch of Gravitron vectors.

When about to fall, $\frac{v^2}{r} = g$. We have $v(r) = 2\pi r/T$ and from part 7.3, $T = 2\pi\sqrt{\frac{R}{a_{\text{c,outside}}}} = 2\pi\sqrt{\frac{R}{3g}}$, so $v(r) = r\sqrt{\frac{3g}{R}}$. From this we can find that $g = \frac{v^2}{r} = \frac{r^2}{r}\frac{3g}{R} = \frac{3g}{rR} \Rightarrow r = \frac{Rg}{3g} = R/3 \Rightarrow \alpha = 1/3$.

Total points: [+7]