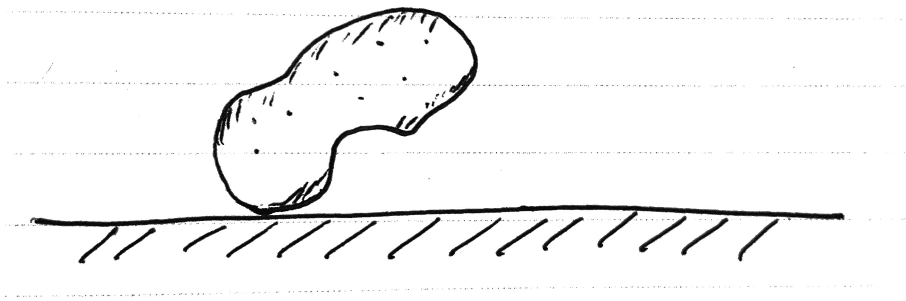
Physics 1A - Winter 2016

Midterm Exam #2

Advice. Your answers will be graded to a large extent on how convincing your reasoning is. A correct answer without good reasoning won't get much credit. Often convincing reasoning is a mixture of mathematics, explanations, and diagrams.

Problem 1. (6 points)

A perfectly smooth but irregularly-shaped rock is placed on a perfectly smooth, planar table on Earth as shown. The coefficient of friction between the rock and the surface is zero. For times $t < 0$, the rock is at rest in an unbalanced orientation, but at $t = 0$ it is released.



Describe the motion of the center of mass of the rock as a function of time for $t > 0$. Justify your description mathematically, with a diagram of the trajectory, and in words.

Solution. There are only two external forces on the rock as it falls: the normal force due to the ground, and gravity. Both of these forces point in the vertical direction. Note, in particular, that there is no friction. Therefore, the net external force on the rock in the x-direction is zero:

$$F_{\text{ext},x} = 0. \tag{1}$$

On the other hand, recall that in general if M is the total mass of a system, and if $a_{\text{cm},x}$ is the x-component of the acceleration of its center of mass, then

$$F_{\text{ext},x} = Ma_{\text{cm},x}. \tag{2}$$

Combining these facts tells us that

$$a_{\text{cm},x} = 0. \tag{3}$$

In other words, the center of mass velocity doesn't change in the x-direction. Since the center of mass started at rest in the x-direction, it will remain at rest in that direction. We therefore conclude that the center of mass falls straight down.

Problem 2. (15 points)

A circuis acrobat of mass M leaps straight up with initial velocity v_0 from a trampoline. As he rises up, he quickly takes a trained monkey of mass m off a perch at height h above the trampoline.

(a) What is the maximum height attained by the pair?

(b) Is mechanical energy of the acrobat-monkey system conserved in this process? If so, prove it. If not, compute the mechanical energy lost.

(c) **Extra Credit. (4 extra points possible)** What would you expect the answer to be in the limits $M/m \to 0$ and $m/M \to 0$? Does you mathematical answer agree with your expectations?

Solution.

(a) The crucial aspect of this scenario to note is that when the acrobat grabs the monkey, we can't be certain that the energy of the system is conserved (in fact we'll see in a moment that it's not), but since this grabbing process is pretty fast, we can apply momentum conservation to the system consisting of the acrobat and monkey just before he grabs the monkey, and just after he grabs the monkey. However, even to apply momentum conservation in this way, we need to know the velocity of the acrobat just before he grabs the monkey. This can be obtained through mechanical energy conservation:

$$\frac{1}{2}Mv_0^2 = Mgh + \frac{1}{2}Mv_b^2 \tag{4}$$

where v_b is the acrobat's velocity right before he grabs the monkey. Momentum conservation applied as described above tells us that

$$Mv_b = (M+m)v_a \tag{5}$$

where v_a is the velocity of the acrobat plus monkey just after the grab. To obtain the maximum height $h_{\max}$ reached by the pair, we can again use energy conservation:

$$(M+m)gh + \frac{1}{2}(M+m)v_a^2 = (M+m)gh_{\max} \tag{6}$$

This is a set of three equations in three unknowns $v_b, v_a, h_{\max}$. Solving for $h_{\max}$ gives

$$\boxed{h_{\max} = h + \frac{M^2}{(M+m)^2}\left(\frac{v_0^2}{2g} - h\right)} \tag{7}$$

(b) Mechanical energy of the system is conserved during the periods in which the acrobat or acrobat + monkey are simply rising under the influence of gravity, but it's not hard to see that it's not conserved during the monkey grab. To see this, note that the kinetic energy of the system right before and after is

$$K_b = \frac{1}{2}Mv_b^2, \qquad K_a = \frac{1}{2}(M+m)v_a^2 \tag{8}$$

The change in mechanical energy is thus

$$K_a - K_b = \frac{1}{2}(M+m)v_a^2 - \frac{1}{2}Mv_b^2 \tag{9}$$

Using the momentum conservation equation from the last part, we see that this becomes

$$K_a - K_b = \frac{1}{2}(M+m)\frac{M^2}{(M+m)^2}v_b^2 - \frac{1}{2}Mv_b^2 \tag{10}$$

$$= \frac{1}{2}Mv_b^2\left(\frac{M}{M+m} - 1\right) \tag{11}$$

Since v_b is not given in the problem, we use eq. (4) to solve for $Mv_b^2/2$ to get the final answer:

$$K_a - K_b = \boxed{\left(\frac{1}{2}Mv_0^2 - Mgh\right)\left(\frac{M}{M+m} - 1\right)} \tag{12}$$

(c) When $M/m \to 0$, the monkey is much more massive than the acrobat, and we'd expect $h_{\text{max}} \to h$, the air wouldn't rise any higher after the grab. When $m/M \to 0$, the monkey is barely there, and we would expect it to have no affect on the acrobat, and he would rise to a height determined by

$$\frac{1}{2}Mv_0^2 = Mgh_{\text{max}} \tag{13}$$

which implies that we expect

$$h_{\text{max}} \to \frac{v_0^2}{2g} \tag{14}$$

in this limit. This is precisely what we find in from our boxed expression in part (a) because the term $M^2/(M+m)^2$ goes to 0 in the first case, and it goes to 1 in the second case. Similar reasoning holds for making sure that our answer to part (b) is sensible in these limits. This analysis is left to the reader as an instructive exercise.

Problem 3. (15 points)

Lonestar is a space rocket engineer attempting to design a rocket that can break the Andromeda rocket speed record. The rocket she designs has mass m_R and can hold a mass of fuel m_F. To break the record, Lonestar needs to launch her rocket from rest. She is considering the following propulsion strategies:

Strategy 1. The fuel is expelled in two mass bursts with the first burst expelling half as much mass as the second and with both bursts having speed u relative to the rocket.

Strategy 2. The fuel is expelled continuously at speed u relative to the rocket.

(a) What will be the final speed of the rocket for strategy 1?

(b) What will be the final speed of the rocket for strategy 2?

(c) If the mass of fuel m_F is much larger than the mass of the rocket m_R, then which strategy will yield a larger final rocket speed?

(d) **Extra Credit. (4 extra points possible)** Use Taylor expansions in the variable $x = m_F/m_R$ to determine the final rocket speed for each strategy when the mass of fuel m_R is much smaller than the mass of the rocket m_R. Which strategy yields a larger final rocket speed in this case?

Solution.

(a) Since the sum of the masses of the two bursts is m_F, and since the first expels half as much mass as the second, the mass of the first burst is $m_F/3$ and the mass of the second burst is $2m_F/3$. Applying momentum conservation to the rocket plus all of its fuel for the first burst, and letting the motion of the rocket be in the positive x-direction, we obtain

$$0 = \frac{1}{3}m_F(-u + v_1) + \left(\frac{2}{3}m_F + m_R\right)(v_1) \tag{15}$$

where v_1 is the velocity of the rocket after the first burst. Applying momentum conservation to the rocket plus its remaining fuel for the second burst gives

$$\left(\frac{2}{3}m_F + m_R\right)v_1 = \frac{2}{3}m_F(-u + v_2) + m_R v_2. \tag{16}$$

where v_2 is the velocity of the rocket after the second burst (which is what we're after). Combining these results gives

$$v_2 = \boxed{\left(\frac{\frac{1}{3}m_F}{m_F + m_R} + \frac{\frac{2}{3}m_F}{\frac{2}{3}m_F + m_R}\right)u}. \tag{17}$$

(b) The mass flow equation for a rocket moving in a single dimension with no external forces and with fuel ejected at a *speed* u is

$$0 = M\frac{dv}{dt} - \frac{dM}{dt}(-u) \tag{18}$$

This yields the following velocity as a function of time if the rocket starts at rest:

$$v(t) = -u\ln\frac{M(t)}{M(0)} \tag{19}$$

When all of the fuel is exhausted, one has $M(t) = m_R$. Also note that $M(0)$, the initial mass of the system, is just $m_R + m_F$. Hence, the velocity of the rocket after all fuel has been depleted is

$$v_{\text{final}} = u\ln\frac{m_R + m_F}{m_R} = \boxed{u\ln\left(1 + \frac{m_F}{m_R}\right)} \tag{20}$$

(c) If m_F is much greater than m_R, then this corresponds to the mathematical limit $m_F/m_R \to \infty$. In this limit, the answer for the first strategy gives $v_2 \to 4u/3$ while for the second strategy $v_{\text{final}} \to \infty$. Therefore the continuous mass flow strategy gives a higher final speed in this limit.

(d) Notice that our answers from both parts can be written in terms of the ratio $x = m_F/m_R$ as follows:

$$v_2 = \left(\frac{\frac{1}{3}x}{1+x} + \frac{\frac{2}{3}x}{1+\frac{2}{3}x}\right)u, \qquad v_{\text{final}} = u\ln(1+x). \tag{21}$$

Taylor expansions of these expressions around $x = 0$ both give have the same first nonzero term that is first-order in x, and the next terms are of order x^2 or high, and can be neglected when x is small:

$$v_2 = xu + O(x^2), \qquad v_{\text{final}} = xu + O(x^2)\cdots \tag{22}$$

Hence, both strategies are equivalent in this limit.

Extra Space

Problem	Score
1	
2	
3	
Total	
Extra Credit 2	
Extra Credit 3	