Lecture 2
Midterm Exam #1 - Solutions

Problem 1.

A ball is thrown into the air at a speed v and an angle θ above the horizontal direction. Air resistance may be neglected.

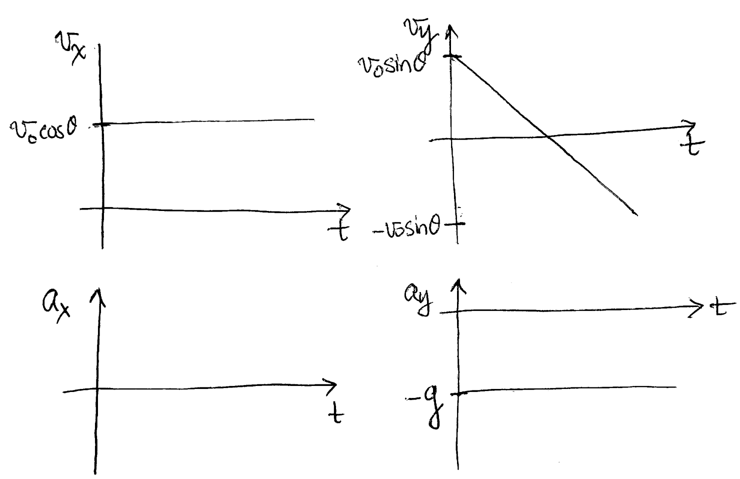
(a) Sketch graphs of the x- and y-components of the velocity and acceleration of the ball as functions of time while it's in the air.

(b) Determine an expression for the angle between the velocity and acceleration vectors of the ball as a function of time while it's in the air.

(c) Use your expression from the last part to determine the angle between the acceleration and velocity vector at the top of its trajectory. Comment on whether your result makes physical sense.

Solution.

(a) We take the positive x-direction to the right and the positive y-direction upward away from the surface of the Earth. Because air resistance is being neglected, there is zero net force in the x-direction on the ball, so $a_x = 0$ and $v_x = v_0 \cos\theta$. In the y-direction, there is only gravity, which causes $a_y = -g$ and $v_y = v_0 \sin\theta - gt$. The graphs of these quantities as functions of time are pictured below.



(b) The angle $\phi(t)$ between the velocity and acceleration vectors as a function of time can be computed using the dot product:

$$\phi(t) = \cos^{-1}\frac{\mathbf{v}(t)\cdot\mathbf{a}(t)}{|\mathbf{v}(t)||\mathbf{a}(t)|} \tag{1}$$

$$= \cos^{-1}\frac{(v_0\cos\theta, v_0\sin\theta - gt)\cdot(0,-g)}{\sqrt{v_0^2\cos^2\theta + v_0^2\sin^2\theta - 2v_0 gt\sin\theta + g^2t^2}\cdot g} \tag{2}$$

$$= \cos^{-1}\boxed{\frac{gt - v_0\sin\theta}{\sqrt{v_0^2 - 2v_0 gt\sin\theta + g^2t^2}}} \tag{3}$$

(c) At the top of the trajectory, the vertical component of the velocity is zero, this implies that $v_0\sin\theta - gt = 0$. It follows that the numerator of the argument of $\cos^{-1}$ in the boxed expression above is zero, so $\phi = \cos^{-1}(0) = \pi/2$ at that moment in time. This makes physical sense because at the top of the trajectory, the velocity vector, which is everywhere tangent to the trajectory, points horizontally, while the acceleration points vertically downward, so these two vectors are perpendicular at that moment.

Problem 2.

Two identically constructed spacecraft in outer space are trying to fly in opposite directions, but they're prevented from doing so by a strong cable that connects them. When they turn on their engines, they are each pulling on opposite ends of the cable with a force of $5\times10^6\,\mathrm{N}$.

(a) If the cable is massless, what is the tension in the middle of the rope?

(b) If the cable has nonzero mass M and length ℓ, then does the answer change? If so, explain why. If not, explain why not.

Solution.

(a) Consider drawing a free body diagram of the left-hand half of the rope. There will be two forces on it: the force $F_{\text{spacecraft}}$ of the spacecraft pulling to the left, and the tension T_{middle} of the right-hand half of the rope pulling to the right. On the other hand, the whole rope, including the left half, has zero acceleration since the two spacecraft are pulling with equal forces on either side. Therefore, applying Newton's Second Law to the left-hand half of the rope gives

$$T_{\text{middle}} - F_{\text{spacecraft}} = M_{\text{left-half}}\,a = 0\cdot 0 = 0. \tag{4}$$

It follows that

$$T_{\text{middle}} = F_{\text{spacecraft}} = \boxed{5\times10^6\,\mathrm{N}} \tag{5}$$

(b) If the cable has nonzero mass M and length ℓ, then this does not change the answer. Even though the mass of the left-half is now nonzero, its acceleration is still zero because the rope as a whole is still not accelerating due to the opposing force of the two spacecraft. It follows that the same Newton's Second Law calculation will hold and the force in the middle will remain unchanged.

Problem 3.

A massless pulley hangs from the ceiling of an elevator by a uniform chain of mass M and length ℓ connected to the pulley's center. Masses m_1 and m_2 are connected by a massless rope and hang on either side of the pulley. The elevator has acceleration A relative to the ground. Let $T(y)$ denote the tension in the chain a distance y above the point at which it is connected to the pulley.

(a) What would you predict $T(0)$ would be if $m_1 = m$, $m_2 = m$ and $A = 0$? Explain using physical reasoning.

(b) Determine the general expression for $T(y)$ in terms of the given variables.

Solution.

(a) When the acceleration of the elevator is zero, and when the masses on both sides are equal, the tension in the rope becomes the same as that of an Atwood's machine with equal masses fixed to a ceiling (strictly speaking in this case the elevator could also have constant velocity, but the same conclusion would apply in that case as well). In that circumstance, the masses would not be accelerating either because they balance one another, and the tension holding the pulley up would simply be the sum of the weights hanging from it, namely

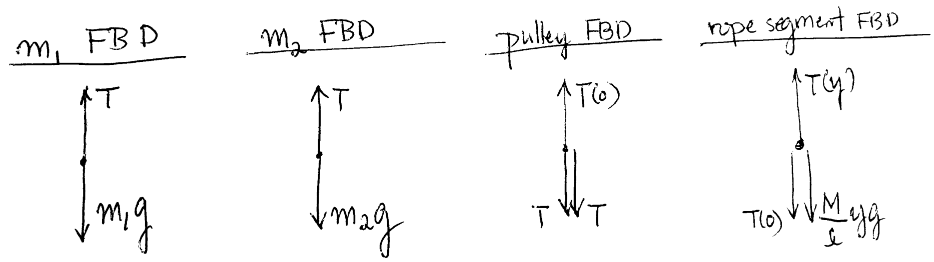
$$\boxed{T(0) = 2mg}. \tag{6}$$

(b) We draw four free body diagrams: one for each mass hanging, one for the pulley, and one for the section of rope between where its is attached to the pulley and a point a distance y above that.

Applying Newton's Second Law to each of these objects in the y-direction (but omitting y subscripts for simplicity) gives

$$T - m_1 g = m_1 a_1, \quad T - m_2 g = m_2 a_2 \quad T(0) - 2T = 0, \quad T(y) - \frac{M}{\ell} y g - T(0) = \frac{M}{\ell} y A \tag{7}$$

This is four equations in five unknowns $a_1, a_2, T, T(0), T(y)$. We have already implicitly used Newton's Third Law as much as we can because we noticed that $T(0)$ is the magnitude of the force of the rope on the pulley and the magnitude of the force of



pulley on the rope. Therefore, what remains is to find an appropriate constraint. Noting that the rope is of constant length, one can systematically obtain the following constraint:

$$2A = a_1 + a_2. \tag{8}$$

We now have five equations in five unknowns, and we can solve. First, let's use the first two Newton's Second Law equations and the constraint to get rid of a_1 and a_2. Dividing the first NSL equation by m_1 and the second by m_2, adding them together, and then invoking the constraint, we obtain

$$T\left(\frac{1}{m_1} + \frac{1}{m_2}\right) - 2g = 2A. \tag{9}$$

Solving for T and then plugging the result into the NSL equation for the pulley gives

$$T(0) = \frac{4m_1 m_2}{m_1 + m_2}(g + A) \tag{10}$$

Finally, we plug this into the NSL equation for the segment of rope and solve for $T(y)$ to obtain the desired result:

$$\boxed{T(y) = \left(\frac{4m_1 m_2}{m_1 + m_2} + \frac{M}{\ell}y\right)(g + A)} \tag{11}$$

If we take the limit $A \to 0$ and set $m_1 = m_2 = m$, then this gives

$$T(0) = \left(\frac{4m \cdot m}{m + m} + 0\right)(g + 0) = 2mg \tag{12}$$

as we had predicted in part (a) on physical grounds.