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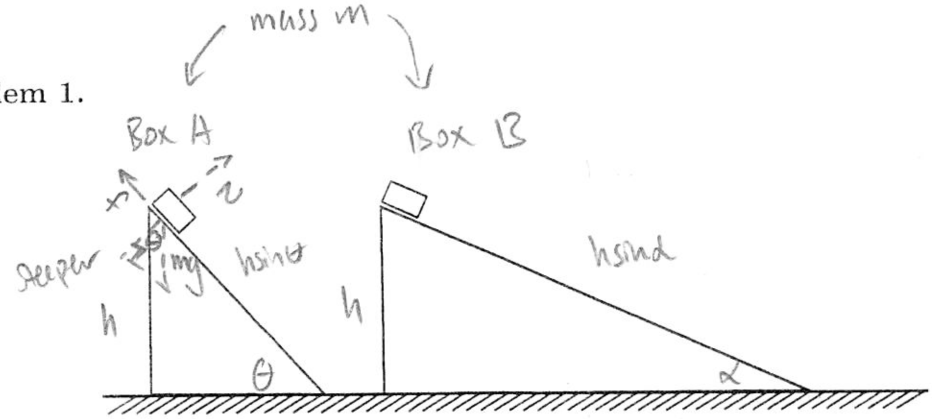
Physics 1A - Winter 2016
Lecture 3

Final Exam

Problem 1.

$\sin\theta = \frac{h}{L}$

$L = h/\sin\theta$



Two identical boxes slide down different ramps pictured above having started at rest from their tops. The boxes start at the same height. The coefficient of kinetic friction μ_k between the boxes and the ramps is the same in both cases, and both ramps are fixed to the ground. The coefficient of static friction is not large enough to prevent the blocks from sliding down the ramps.

Let box A be the box on the left, and let box B be the box on the right. Consider the following statements:

I. The speed of box A is greater than the speed of box B when they reach the bottoms of their ramps.

II. The speed of box A is less than the speed of box B when they reach the bottoms of their ramps.

III. The speed of box A is the same as the speed of box B when they reach the bottoms of their ramps.

Questions.

(a) Use force methods (energy methods not allowed!) to determine which if these statements is true when $\mu_k \neq 0$. You'll find it useful to name your own relevant variables so that you can answer this question mathematically.

(b) Use energy methods to determine which if these statements is true when $\mu_k \neq 0$. You'll find it useful to name your own relevant variables so that you can answer this question mathematically.

(c) Does the answer change when $\mu_k = 0$? Justify using your answers from the previous parts.

a) Box A: $\Sigma F_y = N - mg\cos\theta = 0$

$$N = mg\cos\theta$$

$$\Sigma F_x = mg\sin\theta - \mu_k N = ma_A$$

$$mg\sin\theta - mg\cos\theta \cdot \mu_k = ma_A$$

$$a_A = g\sin\theta - g\cos\theta \cdot \mu_k$$

Box B: $\Sigma F_y = N - mg\cos\alpha = 0$

$$N = mg\cos\alpha$$

$$\Sigma F_x = mg\sin\alpha - \mu_k N = ma_B$$

$$mg\sin\alpha - mg\cos\alpha \cdot \mu_k = ma_B$$

$$g\sin\alpha - g\cos\alpha \cdot \mu_k = a_B$$

Velocity A:

$$V_f^2 = \cancel{V_0^2} + 2a \cdot x$$

$$V_{fA} = \left(2(g\sin\theta - g\cos\theta \cdot \mu_k) \cdot \frac{h}{\sin\theta}\right)^{1/2}$$

Velocity B: $V_f^2 = \cancel{V_0^2} + 2ax$

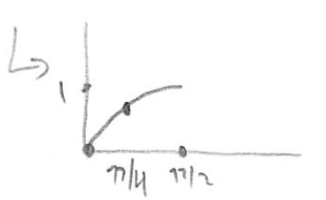
$$V_{fB} = \left(2(g\sin\alpha - g\cos\alpha \cdot \mu_k) \cdot \frac{h}{\sin\alpha}\right)^{1/2}$$

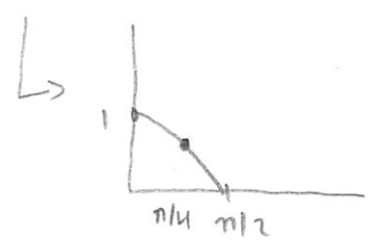
$$\frac{V_{fB}}{V_{fA}} = \sqrt{\frac{2(g\sin\alpha - g\cos\alpha \cdot \mu_k) \cdot h/\sin\alpha}{2(g\sin\theta - g\cos\theta \cdot \mu_k) \cdot h/\sin\theta}}$$

$$\frac{V_{fB}}{V_{fA}} = \sqrt{\frac{(\sin\alpha - \cos\alpha \cdot \mu_k)}{(\sin\theta - \cos\theta \cdot \mu_k} \cdot \frac{1/\sin\alpha}{1/\sin\theta}}$$

Since block A is on a steeper incline, $\theta > \alpha$.

And since $\sin\theta$ and $\cos\theta$





So, we know that $\sin\theta > \sin\alpha$, $\cos\theta < \cos\alpha$. So, the ratio $\frac{V_{fB}}{V_{fA}}$ should be less than 1 since the denominator $\frac{(\sin\theta - \cos\theta \mu_k)}{\sin\theta}$ should be greater than $\frac{(\sin\alpha - \cos\alpha \mu_k)}{\sin\alpha}$ because $\sin\theta - \cos\theta \mu_k > \sin\alpha - \cos\alpha \mu_k$ due to the mathematical statements we made earlier. True, $\sin\alpha < \sin\theta$, but for block B the fraction would be a small number / small number whereas for block A the fraction would be a slightly big number / a number less than 1.

So, velocity block A > velocity block B,

b) block A

$$mgh = \frac{1}{2} m V_A^2 + (\mu_k N) \cdot \frac{h}{\sin\theta}$$
$$gh = \frac{1}{2} V_A^2 + \frac{\mu_k g \cos\theta \cdot h}{\sin\theta}$$

block B

$$mgh = \frac{1}{2} m V_B^2 + \mu_k N \cdot \frac{h}{\sin\alpha}$$
$$gh = \frac{1}{2} V_B^2 + \frac{\mu_k g \cos\alpha}{\sin\alpha}$$

Note that both blocks start with the same potential energy but that block b has a smaller normal force than block A because $\cos\theta < \cos\alpha$. Also, notice that block b traverses through a longer path, because $\sin\alpha < \sin\theta$ so $\frac{h}{\sin\alpha} > \frac{h}{\sin\theta}$, therefore, since block B has a larger normal force and a longer path, the work done by friction on block B > work done by friction on block A, so there is less energy conversion for block B from potential to translational,

So, $V_B < V_a$.

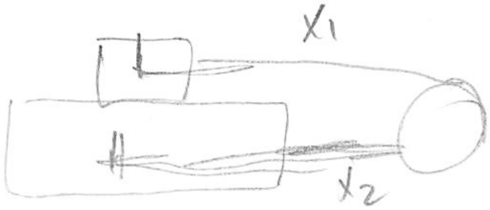
c) Yes the answer changes when $\mu_k = 0$. If the ground is frictionless, then $V_A = V_B$. In part A we can ignore the frictional force and obtain $a_A = g\sin\theta$, $a_B = g\sin\alpha$, but $V_f^2 = 2a x = 2g\sin\theta \cdot \frac{h}{\sin\theta}$ which is the same as $2g\sin\alpha \cdot \frac{h}{\sin\alpha}$.
So, velocities are clearly the same.

in part B we see that we can ignore the work done by friction and obtain

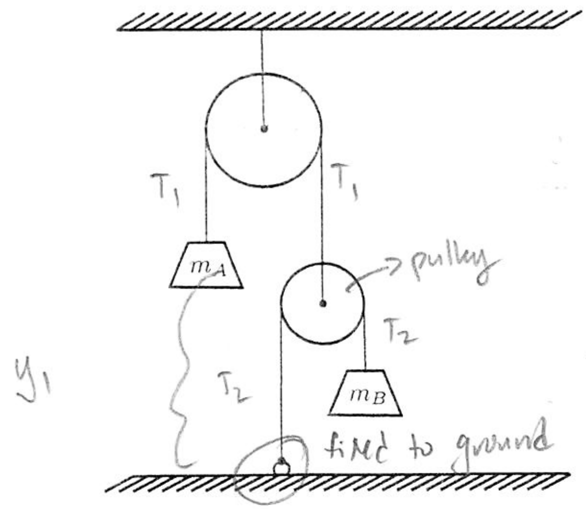
$$mgh = \frac{1}{2} m v_A^2 \quad \text{and} \quad mgh = \frac{1}{2} m v_B^2$$

$$v_A = \sqrt{2gh} \qquad v_B = \sqrt{2gh},$$

so velocities are the same.



Problem 2.



Consider the apparatus above. All pulleys and ropes are massless.

(a) In limit $m_B \to 0$, would you expect the magnitude of the acceleration of mass A to be greater than, equal to, or less than g? Explain using physical reasoning.

(b) Determine an expression for the acceleration of mass A in terms of the given variables.

(c) Determine an expression for the tension in the top rope in terms of the given variables.

(d) Does your mathematical answer in part (b) agree with your answer in part (a)? Explicitly verify this mathematically. If the answers don't agree, you should consider re-evaluating either your math, or your intuition, or both.

a) I would expect $a_{mass\,A} = g$. Since $m_B \to 0$, the entire right side of the system will have a total of 0 mass. So, the tension in the rope connected to A would only be connected to mass A, meaning the tension in the rope will be 0, so $mass_A$ will fall with acceleration g

b) constraint: $A_A = -A_{pulley}$, $T_1 = 2T_2$

mass B $\rightsquigarrow$ $T_2 - m_B g = m_B a_B$

mass A $\rightsquigarrow$ $T_1 - m_A g = m_A A_A$

$\uparrow T$

$\downarrow m_B g$

Also, note that because of the bottom pulley has a mass attached to the ground
length of rope $= y_1 + \pi r + y_1 - y_{m_b} \rightsquigarrow a_{mb} = 2 \cdot A_{pulley}$

Equations: $A_a = -A_{pulley}$

$A_B = 2A_{pulley}$

$T_1 = 2T_2$

$T_2 - m_B g = m_B A_B$

$T_1 - m_A g = m_A A_A$

want to solve for A_A

$\hookrightarrow \dfrac{T_1 - m_A g}{m_A} = A_A$

$$\left.\begin{aligned} T_2 &= m_B A_B + m_B g \\ A_B &= 2A_{pulley} = -2A_a \end{aligned}\right\} T_2 = -2m_B A_A + m_B g$$

$\hookrightarrow 2(-2m_B A_a + m_B g) - m_A g = m_A A_A$

$2m_B g - m_A g = A_A(4m_B + m_a)$

$$\boxed{A_A = \frac{2m_B g - m_A g}{4m_B + m_A}}$$

c) $T_1 = m_A A_A + m_A g$

$$\boxed{T_1 = m_A\left(\frac{2m_B g - m_A g}{4m_B + m_A}\right) + m_A g}$$

d) Yes, the mathematical answer agrees with my physical intuition.

As $m_B \to 0$, $A_A = \dfrac{2(0) - m_A g}{4(0) + m_A} = -\dfrac{m_A g}{m_A} = -g.$

Also, from part C, as $m_B \to 0$, $T_1 = m_A\left(\dfrac{0 - m_A g}{0 + m_A}\right) + m_A g$

$T_1 = -m_A(g) + m_A g = 0.$

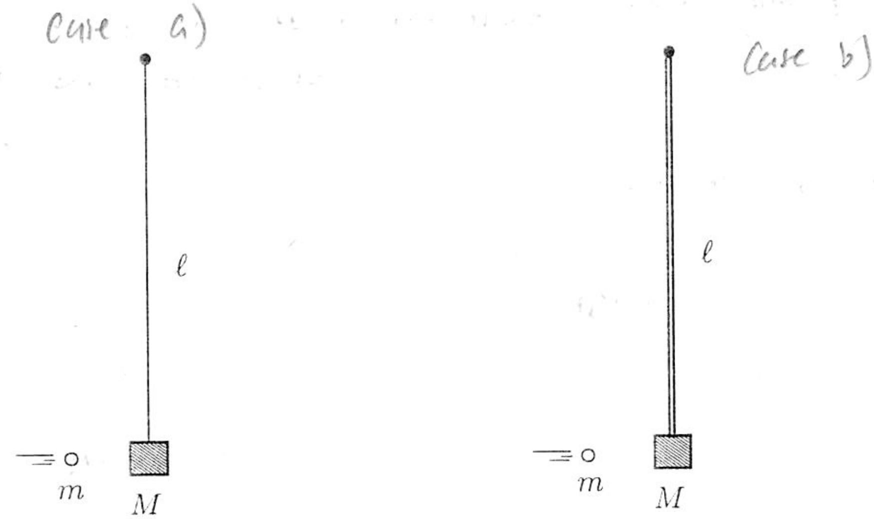
So, both answers in fact agree with my intuition.

Extra Space

Extra Space

Problem 3.

energy
kinematics
momentum
angular momentum
torque
force



In the diagram on the left, a small block of mass M is connected to a massless string of length ℓ that is frictionlessly pivoted at its end so it can spin around in a vertical circle. inextensible In the diagram on the right, an identical block of mass M is connected to a rigid, massless rod of length ℓ that is frictionlessly pivoted at its end so it can spin around in a vertical circle. In both cases, the block starts out hanging at rest, and a clay pellet of mass m is fired horizontally at the block and gets lodged inside.

(a) In which case would you expect the pellet needs to be shot with a higher speed for the block to move all the way around in a vertical circle with radius ℓ?

(b) In the case on the left, with what speed does the pellet need to be shot at the block so the block will make it all the way around in a vertical circle of radius ℓ?

(c) In the case on the right, with what speed does the pellet need to be shot at the block so the block will make it all the way around in a vertical circle of radius ℓ?

(d) According to your answers to parts (b) and (c), in which case does the speed of the pellet need to be greater? Does your mathematics agree with your intuition from part (a)?

a) I believe case A will need to have a pellet shot a higher speed. In case B, the pellet and block system needs to simply just past the maximum vertical height; after reaching this point gravity will simply let the system fall down, thus completing the vertical circle. In case (A) however, the system must maintain a constant centripetal acceleration of $\frac{v^2}{r}$ to complete the circle.

b) Because the collision between the bullet and block is extremely short, we can say that momentum is conserved since the impulse, which is equal to the change of momentum, is 0 because $J = F \cdot time$ when time is close to 0, $\rightsquigarrow \frac{d\bar{P}}{dt} = \Sigma F \cdot dt \approx 0$

So, $P_i = P_f$

$$mV_i = (m+M)V_f$$

After collision, the pair is traveling at velocity V_f.

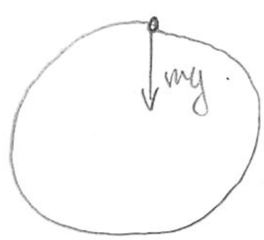
At peak:

$$KE_i = PE_f + KE_f$$

$$\frac{1}{2}(m+M)V_f^2 = 2(m+M)gL + \frac{1}{2}(m+M)V_{centripetal}^2$$

$$\frac{1}{2}V_f^2 = 2gL + \frac{1}{2}V_{centripetal}^2$$

At top of circle.



mg is providing the force required to keep system moving

$$\Rightarrow mg = ma_c = m \cdot \frac{v^2}{r}$$

$$V_{centripetal} = \sqrt{rg}$$

$$\frac{1}{2}V_f^2 = 2gL + \frac{1}{2}\cdot(\sqrt{Lg})^2$$

$$V_f^2 = 4gL + Lg$$

$$V_f = \sqrt{5Lg}$$

$$\Rightarrow V_i = \frac{(m+M)V_f}{m}$$

The bullet must be shot with a velocity of $\boxed{\frac{(m+M)\cdot\sqrt{5LG}}{m}}$

c) In the case on the right, the bullet-block system simply has to reach the top peak of the circle.

So, using the same logic in part (B) for momentum conservation,

$mV_i = (m+M)V_f$ where V_f is the velocity that the pair is moving with.

To reach the top and nothing else,

$$KE_i = PE_f$$

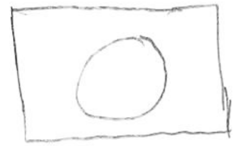
$$\frac{1}{2}(m+M)V_f^2 = (m+M)gH$$

$$V_f = \sqrt{2g \cdot 2L}$$

$$V_i = \frac{(m+M)V_f}{m} = \boxed{\frac{(m+M) \cdot 2\sqrt{gL}}{m}}$$

d) From the answers obtained in parts B and C, the velocity for the non-rigid string case needs to be higher. This result does agree with my intuition since $\frac{(m+M)}{m} \cdot \sqrt{5gL} > \frac{(m+M)}{m} \cdot 2\sqrt{gL}$.

Extra Space



Problem 4.

A large, uniform solid disk of mass M and radius R initially spins on the surface of a flat, frictionless surface at an angular speed ω. Its center of mass is initially at rest relative to the table. Recall that the moment of inertia of a uniform solid disk for rotations about an axis passing through its center of mass and perpendicular to its face is $MR^2/2$.

Josh and Nancy are initially standing diametrically opposite one another on the edge of the disk (so the initial distance between them is $2R$). Next they walk directly toward one another along the diameter joining them until they meet at the disk's center.

They both move with the same speed as a function of time and they both have mass $(c/4)M$ where c is a unitless constant. Let $\mathbf{P}$ denote the total momentum of the Josh + Nancy + disk system. Let $\mathbf{L}_{\parallel}$ denote the angular momentum of the Josh +Nancy + disk system in the direction parallel to the axis of rotation.

(a) Is $\mathbf{P}$ conserved as Josh and Nancy walk to the center?

(b) What is the motion of the center of mass of the disk as Josh and Nancy walk to the center?

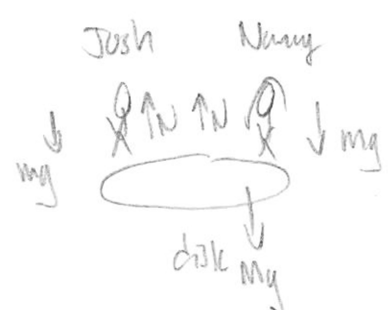
(c) Is $\mathbf{L}_{\parallel}$ conserved as Josh and Nancy walk to the center?

(d) Determine the angular speed of the disk when Josh and Nancy are at its center.

(e) Is the mechanical energy of the system conserved as Josh and Nancy walk to the center? If it is conserved, prove it. If not, compute the change in mechanical energy and show that it's nonzero. In both cases, give physical reasoning to explain why your answer makes sense as well. If it doesn't make sense, you may consider re-evaluating either your intuition about this scenario, or your math, or both.

(f) What would you expect the answer to parts (d) and (e) would be in the limit $c \to 0$? Do your mathematical answers agree with these expectations?

a)



There are no net external forces acting on Josh, Nancy, and the disk, so their momentum is conserved. Also, since all forces are in the z-direction, and $\tau = r \times F$, we know that there are no net torques in the z-direction but there will be in the other directions. So L_z is conserved but L_x and L_y may not be. So total momentum is not conserved.

b) Since the disk was uniformly solid, the center of mass was initially at the center, then because the two people walk with the same speed and have the same mass, and began equidistant from the center, the center of mass does not change as the pair moves toward the center. Also, $F_{ext,net} = 0 = m \cdot a_{cm}$. Since $a_{cm} = 0$, and the disk is at rest relative to table, $V_{cm} = 0 \Rightarrow$ position of center of mass is constant.

c) Yes $L_{//}$ is conserved because of the reason in part A. There are no net external torques in the z-direction because all forces act on the z-direction. So since $\tau = r \times F$, $r \times F$ is not in z-direction, and thus

$$\tau_z = 0 \Rightarrow L_z = L_{\parallel} \text{ is conserved because } \Sigma \tau_z \cdot dt = \frac{dL}{dt}.$$

d) initial angular speed $\Rightarrow \omega$

initial moment of inertia $= \frac{1}{2}MR^2 + 2\left(\frac{c}{4}M\right)R^2$

final moment of inertia $= \frac{1}{2}MR^2$ since both people are at the center

so, $L_{\parallel}$ conserved

$$\Rightarrow \left(\frac{1}{2}MR^2 + \frac{c}{2}MR^2\right)\omega = \frac{1}{2}MR^2 \cdot \omega_{final}$$

$$\omega_{final} = \frac{\left(\frac{1}{2}MR^2 + \frac{c}{2}MR^2\right)\omega}{\frac{1}{2}MR^2}$$

$$\boxed{\omega_{final} = (1 + c)\,\omega}$$

e) Mechanical energy is conserved. As the two people are walking towards the center, they exert equal but opposite forces onto the disk. But because they are exerting equal but opposite forces, the net work done is 0. Moreover, the other forces acting on this system are gravity and normal forces. But the normal force is perpendicular to motion, so work due to normal forces $= 0$. As such, since the work of nonconservative forces is 0, mechanical energy must be conserved.

f) As $c \to 0$, the two people become massless, so angular velocity should remain unchanged, and mechanical energy should be the same as well. In fact our answers support this.

In part (D) ~> $W_{final} = (1+c)W_0$

$W_{final} = W_0.$

In part (e) ~> as $c \to 0$

mechanical energy is still conserved.

Extra Space

Problem 5.

(a) Consider a projectile launched at speed v angle θ relative to the horizontal on flat ground near the surface of a planet with gravitational acceleration g_P. Derive an expression for the range r_P of the projectile on this planet.

(b) If an object of mass m is a distance r away from the center of a planet of mass M, then it experiences an attractive gravitational force of magnitude

$$F = \frac{GMm}{r^2} \tag{1}$$

where G is Newton's gravitational constant. Using Newton's Second Law to set this equal to the object's mass times the magnitude a of its acceleration, we find that the object's gravitational acceleration is independent of its mass, but depends on G, M and r:

$$a = \frac{GM}{r^2} \tag{2}$$

In other words, it depends only on a fundamental physical constant, the mass of the planet, and the distance to the planet's center. If the object is at a height h above the planet's surface, and if the planet's radius is R, then the gravitational acceleration becomes

$$a = \frac{GM}{(R+h)^2} = \frac{GM}{R^2}\left(1 + \frac{h}{R}\right)^{-2} \tag{3}$$

$$a' = -\frac{2GM}{R^2}\left(1 + \frac{h}{R}\right)^{-3}$$

What is the Taylor expansion of the acceleration due to gravity in the variable $x = h/R$ about $x = 0$ including only the first three nonzero terms?

non-zero order

Useful observations. You should find that the first non-vanishing order equals GM/R^2. When you are close to the surface of the planet, only this first nonzero term is significant because $x = h/R$ will be extremely small, so this expression gives the acceleration due to gravity near the planet's surface.

$$g_P = \frac{GM}{R^2}. \tag{4}$$

In the case of Earth, one can for example show that by using this term to compute the acceleration due to gravity, the quantity GM_E/R_E^2 gives a value very close to g, where M_E is the Earth's mass, and R_E is the Earth's radius. In other words, one can predict the acceleration due to gravity near the Earth's surface using its mass and its radius!

(c) Alice is on planet A whose mass is $M_E/2$ and whose radius is R_E. Bob is on planet B whose mass is $\sqrt{3}M_E$ and whose radius is $\sqrt{2}R_E$.

Alice throws a ball at a speed v at a certain angle relative to the ground that maximizes the range of the thrown object. Bob throws a ball at speed $\sqrt{2}v$ and at an angle θ relative to the ground, and the ball ends up having the same range as Alice's ball.

At what angle θ did Bob throw the ball?

a) $V_{fy} = 0 = V_{oy} + a \cdot t$

$$-V_{oy} = -g_P t$$

$$V\sin\theta = g_P t$$

$$t = \frac{V\sin\theta}{g_P}$$

since it will take $2t$ for the projectile to reach the ground again, and we know that the distance traveled $= x = V_{ox} \cdot t$,

$$\text{range} = V\cos\theta \cdot \frac{2V\sin\theta}{g_P} = \frac{2V^2\cos\theta\sin\theta}{g_P}$$

b) Taylor Expansion

$$f(x) = f(a) + \frac{f'(a)(x-a)}{1!} + \frac{f''(a)(x-a)}{2!} + \dots$$

$$a_{\text{gravitational}} = \frac{GM}{(R+h)^2} = \frac{GM}{R^2}\left(1+\frac{h}{r}\right)^{-2}$$

↳ treat a as a function of $\frac{h}{r}$

$$a' = \frac{-2GM}{R^2}\left(1+\frac{h}{r}\right)^{-3}$$

$$a'' = \frac{6GM}{R^2}\left(1+\frac{h}{r}\right)^{-4}$$

Taylor Expansion about $x=0$ yields

$$\frac{GM}{R^2} - \frac{2GM}{R^2}(x-0) + \frac{6GM}{R^2}\frac{(x-0)^2}{2} = \boxed{\frac{GM}{R^2}\left(1-2x+3x^2\right)}$$

Extra Space

c) planet A $\rightarrow$ mass $\frac{M_E}{2}$

radius R_E

gravity on planet A $= \frac{G\frac{M_E}{2}}{R_E^2} = \frac{1}{2}g$

planet B $\rightarrow$ mass $\sqrt{3}\, M_E$

radius $\sqrt{2}\, R_E$

gravity on planet B $= \frac{G \cdot \sqrt{3} M_E}{2 R_E^2} = \frac{\sqrt{3}}{2} g$

gravity on earth $= g = \frac{GM_E}{R_E^2}$

Alice: 

$$\text{range} = \frac{2v^2 \sin\theta\cos\theta}{\frac{1}{2}g} = \frac{4v^2\sin\theta\cos\theta}{g} = \frac{2v^2\sin(2\theta)}{g}$$

Bob:

$$\text{range} = \frac{4v^2\sin\alpha\cos\alpha}{\frac{\sqrt{3}}{2}g} = \frac{8v^2\sin\alpha\cos\alpha}{\sqrt{3}g} = \frac{4v^2\sin(2\alpha)}{\sqrt{3}g}$$

$$\frac{4v^2\sin(2\alpha)}{\sqrt{3}\,g} = \frac{2v^2\sin(2\theta)}{g}$$

$$\sin(2\alpha) = \frac{\sqrt{3}}{2}\sin(2\theta)$$

To maximize range, however, Alice must have thrown the ball at $\theta = \pi/4$

$$\Rightarrow \quad x = \frac{2v^2\cos\theta\sin\theta}{g}$$

$$x' = \frac{2v^2}{g}\left(-\sin^2\theta + \cos^2\theta\right) = 0$$

$x' \Rightarrow \sin^2\theta = \cos^2\theta \Rightarrow \theta = \pi/4$ to maximize

range

So, $\sin(2\alpha) = \frac{\sqrt{3}}{2}\sin\left(\frac{\pi}{2}\right)$

$\sin(2\alpha) = \frac{\sqrt{3}}{2} \Rightarrow 2\alpha = \frac{\pi}{3} \Rightarrow \boxed{\alpha = \frac{\pi}{6}}$

Extra Space

Extra Space