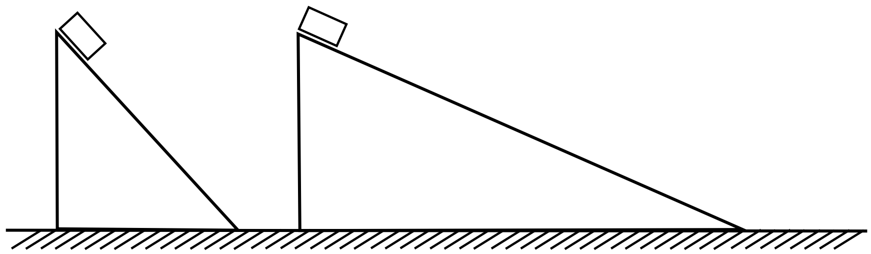
Physics 1A - Winter 2016

Lecture 3

Final Exam

Problem 1.



Two identical boxes slide down different ramps pictured above having started at rest from their tops. The boxes start at the same height. The coefficient of kinetic friction μ_k between the boxes and the ramps is the same in both cases, and both ramps are fixed to the ground. The coefficient of static friction is not large enough to prevent the blocks from sliding down the ramps.

Let box A be the box on the left, and let box B be the box on the right. Consider the following statements:

I. The speed of box A is greater than the speed of box B when they reach the bottoms of their ramps.

II. The speed of box A is less than the speed of box B when they reach the bottoms of their ramps.

III. The speed of box A is the same as the speed of box B when they reach the bottoms of their ramps.

Questions.

(a) Use force methods (energy methods not allowed!) to determine which if these statements is true when $\mu_k \neq 0$. You'll find it useful to name your own relevant variables so that you can answer this question mathematically.

(b) Use energy methods to determine which if these statements is true when $\mu_k \neq 0$. You'll find it useful to name your own relevant variables so that you can answer this question mathematically.

(c) Does the answer change when $\mu_k = 0$? Justify using your answers from the previous parts.

Solution.

(a) We'll determine the speed of the box at the bottom of a ramp of height h and whose surface makes an angle θ with the horizontal, then we'll see how the speed varies as a function of θ. Box A slides down a ramp with a larger value of θ, so we should be able use the functional dependence of the speed at the bottom on theta to determine if it reaches the bottom of the ramp with a speed that is greater than or less than that of box B. Our strategy for finding the speed at the bottom using forces will be to use NSL to determine the acceleration of the box in the direction parallel to the surface of the ramp, and then use kinematics to determine its speed at the bottom.

If we draw a free body diagram for a box on a ramp of height h and angle θ, and if we orient our axes so that the positive x-axis points down the ramp, while the positive y-axis points upward, perpendicular to the ramp, we obtain the following equations from NSL in the x- and y-directions:

$$mg\sin\theta - f_k = ma_x, \qquad N - mg\cos\theta = 0 \tag{1}$$

where f_k is the magnitude of the force of kinetic friction. We also have the following relationship between the magnitude of the friction force and the normal force:

$$f_k = \mu_k N. \tag{2}$$

This is a system of three equations in three unknowns f_k, N, a_x. We only really want to solve for a_x:

$$a_x = (\sin\theta - \mu_k\cos\theta)g. \tag{3}$$

The kinematics equation $v_{f,x}^2 = v_{i,x}^2 + 2a_x(x_f - f_i)$ then allows us to solve for the speed at the bottom of the ramp. The length of the part on which the box slides in terms of the height h and angle θ of the ramp is $x_f - x_i = h/\sin\theta$. Therefore we have

$$v_{f,x} = \sqrt{2(\sin\theta - \mu_k\cos\theta)g\frac{h}{\sin\theta}} = \sqrt{2gh}\sqrt{1 - \frac{\mu_k}{\tan\theta}} \tag{4}$$

Notice that since $\tan\theta$ is an increasing function of θ, the expression $\mu_k/\tan\theta$ is a decreasing function of θ, so the expression $1 - \mu_k/\tan\theta$ is an increasing function of θ. Therefore, the speed at the bottom of the ramp is greater for larger θ. So statement I is true.

(b) We reproduce the expression for $v_{f,x}$ using energy methods, then the rest of the reasoning from part (a) carries over unscathed. We set the zero of potential energy to be the bottom of the ramp, then the potential energy at the top equals the kinetic energy at the bottom *minus* the work done by friction since friction is the only non-conservative force that performs nonzero work:

$$mgh = \frac{1}{2}mv_f^2 - \left(-\mu_k mg\cos\theta\frac{h}{\sin\theta}\right) \tag{5}$$

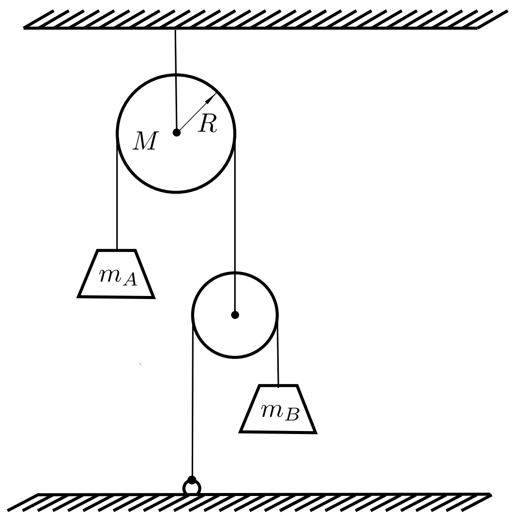
solving for v_f gives precisely the expression obtained through force methods in part (a):

$$v_f = \sqrt{2gh}\sqrt{1 - \frac{\mu_k}{\tan\theta}} \tag{6}$$

Well aint that nifty!

(c) When there is no friction, $\mu_k \to 0$, and the expression for v_f reduces to $\sqrt{2gh}$, an expression independent of θ. This makes sense because when there is no friction, the kinetic energy of the block at the bottom is the same as the potential energy at the top, regardless of the ramp's length.

Problem 2.



Consider the apparatus above. All pulleys and ropes are massless.

(a) In limit $m_B \to 0$, would you expect the magnitude of the acceleration of mass A to be greater than, equal to, or less than g? Explain using physical reasoning.

(b) Determine an expression for the acceleration of mass A in terms of the given variables.

(c) Determine an expression for the tension in the top rope in terms of the given variables.

(d) Does your mathematical answer in part (b) agree with your answer in part (a)? Explicitly verify this mathematically. If the answers don't agree, you should consider re-evaluating either your math, or your intuition, or both.

Solution.

(a) In the limit $m_B \to 0$, the situation effectively reduces to mass m_A suspended from a rope that is wrapped around the pulley. Since the pulley is massless, it's rotation will not impede the motion of mass m_A, and we would expect it to fall freely – the magnitude of its acceleration should be equal to g.

(b) Applying Newton's Second Law in the y-direction (which we take upward positive) to mass A, mass B, and the pulley from which mass B hands, we find the following Newton's Second Law equations:

$$T_A - m_A g = m_A a_{A,y}, \qquad T_B - m_B g = m_B a_{B,y}, \qquad T_A - 2T_B = 0. \tag{7}$$

This is a system of three equations in four unknowns $T_A, T_B, a_{A,y}, a_{B,y}$. We are missing an equation. As always, when we're at this point in such a problem, we should look for some constraints. It should be pretty clear from the diagram that the accelerations of the two masses are related somehow, but how exactly? By drawing one of our standard diagrams for deriving constraints, we find the following relationships between the positions y_A and y_B of masses A and B, the positions $y_{P,A}$ and $y_{P,B}$ of the pulleys from which they hang respectively, the radii R_A and R_B of those pulleys, and the lengths ℓ_A and ℓ_B of the ropes from which masses A and B hang. We take the ground to have position $y = 0$.

$$y_{P,A} - y_A + \pi R_A + y_{P,A} - y_{P,B} = \ell_A \tag{8}$$

$$y_{P,B} - 0 + \pi R_B + y_{P,B} - y_B = \ell_B. \tag{9}$$

Taking two time derivatives on both sides of these equations, and taking note of the fact that the positions of the upper pulley, the radii of both pulleys, and the lengths of both ropes are constant, we find

$$-a_{A,y} - a_{P,B} = 0 \tag{10}$$

$$2a_{P,B} - a_{B,y} = 0. \tag{11}$$

Combining these equations gives the desired constraint relating the accelerations of the two masses:

$$a_{B,y} = -2a_{A,y}. \tag{12}$$

Now we're cookin'! Combining this with our NSL equations gives four equations in four unknowns, and we can solve. Well I dunno about you, but I'm pumped to do some algebra. Using the last NSL equation to eliminate T_B in favor of T_A, and using the constraint to eliminate $a_{B,y}$ in favor of $a_{A,y}$, we obtain the following system of two equations in two unknowns for T_A and $a_{A,y}$:

$$T_A - m_A g = m_A a_{A,y}, \qquad \frac{1}{2}T_A - m_B g = m_B(-2a_{A,y}). \tag{13}$$

If we subtract twice the second equation from the first, then we eliminate T_A and obtain the following equation for $a_{A,y}$:

$$-m_A g + 2m_B g = m_A a_{A,y} + 4m_B a_{A,y}, \tag{14}$$

and therefore:

$$a_{A,y} = \boxed{\frac{2m_B - m_A}{4m_B + m_A} g}. \tag{15}$$

(c) Using the same system of two equations in two unknowns from the last steps in part (b), we can easily now compute the tension in the rope holding mass A since we already know its acceleration:

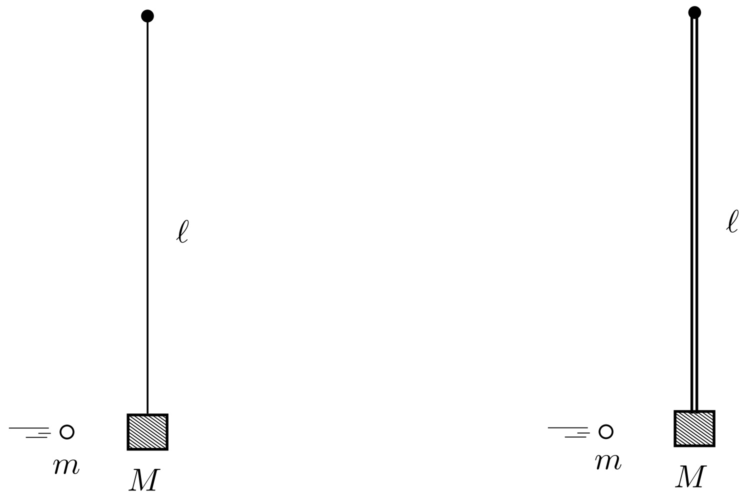
$$T_A = m_A(g + a_{A,y}) = m_A \left(1 + \frac{2m_B - m_A}{4m_B + m_A}\right) g = \frac{6m_A m_B}{4m_B + m_A} \tag{16}$$

As specified on the exam, the "top rope" refers to the rope holding up the top pulley. Applying NSL to this pulley shows that the tension T in this rope is twice T_A. So we have

$$T = \boxed{\frac{12m_A m_B}{4m_B + m_A}} \tag{17}$$

(d) When $m_B \to 0$, we find that $a_{A,y} \to -(m_A/m_A)g = -g$ as described in our prediction above.

Problem 3.



In the diagram on the left, a small block of mass M is connected to a massless string of length ℓ that is frictionlessly pivoted at its end so it can spin around in a vertical circle. In the diagram on the right, an identical block of mass M is connected to a rigid, massless rod of length ℓ that is frictionlessly pivoted at its end so it can spin around in a vertical circle. In both cases, the block starts out hanging at rest, and a clay pellet of mass m is fired horizontally at the block and gets lodged inside.

(a) In which case would you expect the pellet needs to be shot with a higher speed for the block to move all the way around in a vertical circle with radius ℓ?

(b) In the case on the left, with what speed does the pellet need to be shot at the block so the block will make it all the way around in a vertical circle of radius ℓ?

(c) In the case on the right, with what speed does the pellet need to be shot at the block so the block will make it all the way around in a vertical circle of radius ℓ?

(d) According to your answers to parts (b) and (c), in which case does the speed of the pellet need to be greater? Does your mathematics agree with your intuition from part (a)?

Solution.

(a) The block attached to the rod can make it all the way around the circle by just barely having a nonzero velocity at the top, but if the block attached to the string had such a low velocity at the top, then it would fall under the influence of gravity with no rigid rod to keep it up, so I'd expect that the block attached to the string would need a larger initial speed at the bottom, and therefore the pellet would need to strike the block with a higher speed.

(b) We take the positive x-direction to the right and the positive y-direction upward. When the pellet strikes the block attached to the string, we can apply momentum conservation in the x-direction to obtain a relationship between the velocity v_{bottom} of the combined pellet + block system directly after the collision to the velocity v of the pellet before the collision in the x-direction.

$$mv = (m+M)v_{\text{bottom}} \tag{18}$$

We can then use mechanical energy conservation to relate v_{bottom} to v_{top}, the velocity of the pellet + block at the top of its trajectory:

$$\frac{1}{2}(m+M)v_{\text{bottom}}^2 = \frac{1}{2}(m+M)v_{\text{top}}^2 + (m+M)g(2\ell). \tag{19}$$

Finally, if we want the pellet + block system to just make it around the circle, then this corresponds to the string just barely going slack at the top. This means that only the force due to gravity will apply to the system at the top, and NSL in the y-direction at that moment gives

$$-(m+M)g = -(m+M)\frac{v_{\text{top}}^2}{\ell}. \tag{20}$$

We thus have a system in three equations in three unknowns $v, v_{\text{bottom}}, v_{\text{top}}$. Solving for v gives

$$v = \boxed{\frac{m+M}{m}\sqrt{5gl}}. \tag{21}$$

(c) The case on the right which includes the rod instead of the string is different in the respect that because the rod is rigid, it could potentially exert an external force on the pellet + block system during the collision and change its momentum. We can get around this by using angular momentum conservation in the z-direction instead, which, taking the point at which the rod is fixed as the axis of rotation, gives

$$mv\ell = (m+M)v_{\text{bottom}}\ell \tag{22}$$

This case is also different in the respect that the condition for the block to make it around the circle is different. In fact, the velocity of the block at the top can just barely be zero, and it will still make it around the circle. Therefore, the energy conservation equation should read

$$\frac{1}{2}(m+M)v_{\text{bottom}}^2 = (m+M)g(2\ell) \tag{23}$$

We now have a system of two equations in two unknowns v, v_{bottom}. Solving for v gives

$$v = \boxed{\frac{m+M}{m}\sqrt{4g\ell}}. \tag{24}$$

(d) We see from our answers that the speed of the pellet needs to be greater in the case with the string since $\sqrt{5} > \sqrt{4}$. This aligns with out intuition from part (a).

Problem 4.

A large, uniform solid disk of mass M and radius R initially spins on the surface of a flat, frictionless surface at an angular speed ω. Its center of mass is initially at rest relative to the table. Recall that the moment of inertia of a uniform solid disk for rotations about an axis passing through its center of mass and perpendicular to its face is $MR^2/2$.

Josh and Nancy are initially standing diametrically opposite one another on the edge of the disk (so the initial distance between them is $2R$). Next they walk directly toward one another along the diameter joining them until they meet at the disk's center.

They both move with the same speed as a function of time and they both have mass $(c/4)M$ where c is a unitless constant. Let $\mathbf{P}$ denote the total momentum of the Josh + Nancy + disk system. Let $\mathbf{L}_{\parallel}$ denote the angular momentum of the Josh +Nancy + disk system in the direction parallel to the axis of rotation.

(a) Is $\mathbf{P}$ conserved as Josh and Nancy walk to the center?

(b) What is the motion of the center of mass of the disk as Josh and Nancy walk to the center?

(c) Is $\mathbf{L}_{\parallel}$ conserved as Josh and Nancy walk to the center?

(d) Determine the angular speed of the disk when Josh and Nancy are at its center.

(e) Is the mechanical energy of the system conserved as Josh and Nancy walk to the center? If it is conserved, prove it. If not, compute the change in mechanical energy and show that it's nonzero. In both cases, give physical reasoning to explain why your answer makes sense as well. If it doesn't make sense, you may consider re-evaluating either your intuition about this scenario, or your math, or both.

(f) What would you expect the answer to parts (d) and (e) would be in the limit $c \to 0$? Do your mathematical answers agree with these expectations?

Solution.

(a) Since there is no friction on the surface, the net external force on the system in the horizontal direction, the total momentum of the system in that direction is certainly conserved. Moreover, as long as Nancy and Josh walk in such a way that their centers of mass don't accelerate appreciably in the vertical direction, the vertical position of the center of mass of the system remains stationary, so the total momentum of the system in that direction is conserved as well.

(b) Since the total momentum of the system is conserved, the acceleration of the center of mass is zero as Nancy and Josh walk. Since the center of mass starts at rest, it therefore stays at rest.

(c) We take the plane of rotation of the disk to be the x-y plane. The only external forces on the system are gravity and the normal force, both of which are in the vertical direction. Therefore, the torques they exert can only be in the x-y plane. As a result, the net external torque in the z direction is zero, so the angular momentum in that direction, which is precisely $\mathbf{L}_{\|}$, is conserved.

(d) We use angular momentum conservation in the vertical direction, taking the axis of rotation of the disk as the location of the origin. The angular momentum of the system before Nancy and Josh have started walking is therefore

$$L_{z,i} = \left(\frac{c}{4}MR^2 + \frac{c}{4}MR^2 + \frac{1}{2}MR^2\right)\omega \tag{25}$$

when Nancy and Josh are at the center of the disk, the final angular momentum is

$$L_{z,f} = \frac{1}{2}MR^2\omega_f \tag{26}$$

Setting $L_{z,i} = L_{z,f}$ and solving for ω_f gives

$$\omega_f = \boxed{(c+1)\omega} \tag{27}$$

(e) The initial and final gravitational potential energies of the system are the same since no object moves vertically as Josh and Nancy walk from the edge to the center, but the initial and final kinetic energies may not be the same. The initial kinetic energy is

$$K_i = \frac{1}{2}\left(\frac{c}{4}MR^2 + \frac{c}{4}MR^2 + \frac{1}{2}MR^2\right)\omega^2 = \frac{1}{2}\frac{c+1}{2}MR^2\omega^2 \tag{28}$$

the final kinetic energy is

$$K_f = \frac{1}{2}\left(\frac{1}{2}MR^2\right)((c+1)\omega)^2 = (c+1)K_i \tag{29}$$

The difference is therefore

$$K_f - K_i = (c+1)K_i - K_i = cK_i > 0 \tag{30}$$

The kinetic energy of the system increased! Does this make physical sense? Yes. In order for Josh and Nancy to walk to the center, their bodies need to expend some potential energy from the food they've presumably eaten earlier in the day, and this energy shows up in the end as an increase in the kinetic energy of the system.

(f) When $c \to 0$, this corresponds to the masses of Josh and Nancy vanishing. In this case, one would expect that their walking to the center has no affect on the angular speed of the disk, and therefore no effect on the kinetic energy of the system as well. So one would expect that $\omega_f \to \omega$ and $K_f - K_i \to 0$. This is precisely what we find in our mathematical answers since $\omega_f = (c+1)\omega \to \omega$ and $K_f - K_i = cK_i \to 0$.

Problem 5.

(a) Consider a projectile launched at speed v angle θ relative to the horizontal on flat ground near the surface of a planet with gravitational acceleration g_P. Derive an expression for the range r_P of the projectile on this planet.

(b) If an object of mass m is a distance r away from the center of a planet of mass M, then it experiences an attractive gravitational force of magnitude

$$F = \frac{GMm}{r^2} \tag{31}$$

where G is Newton's gravitational constant. Using Newton's Second Law to set this equal to the object's mass times the magnitude a of its acceleration, we find that the object's gravitational acceleration is independent of its mass, but depends on G, M and r:

$$a = \frac{GM}{r^2} \tag{32}$$

In other words, it depends only on a fundamental physical constant, the mass of the planet, and the distance to the planet's center. If the object is at a height h above the planet's surface, and if the planet's radius is R, then the gravitational acceleration becomes

$$a = \frac{GM}{(R+h)^2} = \frac{GM}{R^2}\left(1 + \frac{h}{R}\right)^{-2} \tag{33}$$

What is the Taylor expansion of the acceleration due to gravity in the variable $x = h/R$ about $x = 0$ including only the first three nonzero terms?

Useful observations. You should find that the first non-vanishing order equals GM/R^2. When you are close to the surface of the planet, only this first nonzero term is significant because $x = h/R$ will be extremely small, so this expression gives the acceleration due to gravity near the planet's surface.

$$g_P = \frac{GM}{R^2}. \tag{34}$$

In the case of Earth, one can for example show that by using this term to compute the acceleration due to gravity, the quantity GM_E/R_E^2 gives a value very close to g, where M_E is the Earth's mass, and R_E is the Earth's radius. In other words, one can predict the acceleration due to gravity near the Earth's surface using its mass and its radius!

(c) Alice is on planet A whose mass is $M_E/2$ and whose radius is R_E. Bob is on planet B whose mass is $\sqrt{3}M_E$ and whose radius is $\sqrt{2}R_E$.

Alice throws a ball at a speed v at a certain angle relative to the ground that maximizes the range of the thrown object. Bob throws a ball at speed $\sqrt{2}v$ and at an angle θ relative to the ground, and the ball ends up having the same range as Alice's ball.

At what angle θ did Bob throw the ball?

Solution.

(a) The x- and y-positions of a projectile as a function of time are

$$x = v\cos\theta t, \qquad y = v\sin\theta t - \frac{1}{2}g_P t^2 \tag{35}$$

The projectile strikes the ground when $y = 0$. Using this condition in the y-equation gives a unique nonzero time which is the time at which the projectile strikes the ground after having flown along a parabola:

$$t = \frac{2}{g_P}v\sin\theta. \tag{36}$$

Plugging this into the x-equation gives the desired expression for the range:

$$r_P = \frac{v^2}{g_P}(2\cos\theta\sin\theta) = \boxed{\frac{v^2}{g}\sin(2\theta)}. \tag{37}$$

(b) Using the definition $x = h/R$, we have

$$a = \frac{GM}{R^2}(1+x)^{-2} \tag{38}$$

To determine the Taylor expansion of this expression about $x = 0$, we notice that

$$(1+x)^{-2}\Big|_{x=0} = 1 \tag{39}$$

$$\frac{d}{dx}(1+x)^{-2}\Big|_{x=0} = -2 \tag{40}$$

$$\frac{d^2}{dx^2}(1+x)^{-2}\Big|_{x=0} = 6 \tag{41}$$

so by Taylor's formula we have

$$a = \frac{GM}{R^2}\left(\frac{1}{0!} + \frac{(-2)x}{1!} + \frac{6x^2}{2!} + \cdots\right) = \boxed{\frac{GM}{R^2}(1 - 2x + 3x^2 + \cdots)}. \tag{42}$$

(c) Using the range formula from part (a), we can compare the ranges of the balls thrown by Alice and Bob:

$$\frac{r_A}{r_B} = \frac{(v_A^2/g_A)\sin(2\theta_A)}{(v_B^2/g_B)\sin(2\theta_B)} = \frac{v_A^2}{v_B^2}\frac{g_B}{g_A}\frac{\sin(2\theta_A)}{\sin(2\theta_B)} \tag{43}$$

Since the Alice throws at a range-maximizing angle, we have $\theta_A = \pi/4$ so $\sin(2\theta_A) = 1$. Since the ranges are the same, $r_A/r_B = 1$. We also see from the information in the problem that

$$\frac{v_A^2}{v_B^2} = \frac{v^2}{2v^2} = \frac{1}{2}. \tag{44}$$

and

$$\frac{g_B}{g_A} = \frac{GM_B/R_B^2}{GM_A/R_A^2} = \frac{M_B}{M_A}\frac{R_A^2}{R_B^2} = \frac{\sqrt{3}M_E}{M_E/2}\frac{R_E^2}{2R_E^2} = \frac{2\sqrt{3}}{2} \tag{45}$$

Putting all of these facts together gives

$$1 = \frac{1}{2}\frac{2\sqrt{3}}{2}\frac{1}{\sin(2\theta_B)} \tag{46}$$

Hence, we find that

$$\sin(2\theta_B) = \frac{\sqrt{3}}{2} \tag{47}$$

which implies $2\theta_B = \pi/3$ and therefore $\theta_B = \boxed{\pi/6}$.