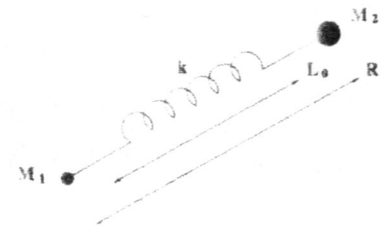
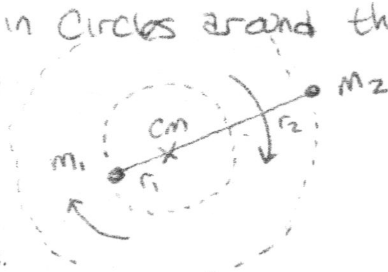


1) In the deepest, darkest reaches of space, far, far away from the influence of other matter and matters of civilization, in a dreadfully isolated and desolate spot known as 'Fresno', two small objects of mass m_1 and m_2 joined by a spring of constant k and natural (unstretched) length L_0, orbit about their mutual center-of-mass with a fixed separation (point-mass-to-point-mass) R in some inertial frame of reference.

- 1a) (5 points) Describe the physics of the situation - in particular, how do we know that the objects orbit around their mutual center-of-mass? Describe the orbital motion of the objects - how does the position of each object relate to that of the other object and the center-of-mass? A quick sketch might help.

$\Sigma \vec{F}_{ext} = 0$, So $\vec{A}_{cm} = 0$, which means $\vec{V}_{cm} =$ Constant. A reference frame attached to the center-of-mass would (in this case) be an inertial frame. In the center-of-mass frame, the center of mass (located directly between the masses) is a fixed point... the masses orbit in circles around the center-of mass, one on either side...



- 1b) (5 points) Find the orbital radius for each object.

$$x_{cm} = \frac{\Sigma m_i x_i}{\Sigma m_i}$$

$$r_1 = \frac{m_1(0) + M_2(R)}{M_1 + M_2}$$

$$r_2 = R - r_1 \quad \ldots>$$

$$\boxed{r_1 = R\frac{M_2}{M_1+M_2} \qquad r_2 = R\frac{M_1}{M_1+M_2}}$$

- 1c) (5 points) With what angular velocity do the objects orbit about the center of mass?

$$\Sigma F_r = ma_r \quad \swarrow (V = r\omega)$$

$$K(R-L_0) = m_1 \frac{V_1^2}{r_1}$$

$$K(R-L_0) = m_1 r_1 \omega^2$$

$$K(R-L_0) = R\frac{M_1 M_2}{M_1+M_2}\omega^2$$

$$\boxed{\omega = \sqrt{\frac{(m_1+M_2)K(R-L_0)}{M_1 M_2 R}}}$$

- (d) (10 points) Find the mechanical energy of the system in this inertial frame of reference as a function of the separation distance between the objects.

$$E = K + U$$

$$E = \frac{1}{2} m_1 V_1^2 + \frac{1}{2} m_2 V_2^2 + \frac{1}{2} k (R - L_o)^2$$

$$E = \frac{1}{2} (m_1 r_1^2 + m_2 r_2^2) \omega^2 + \frac{1}{2} k (R - L_o)^2$$

$$E = \frac{1}{2} \frac{M_1 M_2}{M_1 + M_2} R^2 \omega^2 + \frac{1}{2} k (R - L_o)^2$$

$$E = \frac{1}{2} k R (R - L_o) + \frac{1}{2} k (R - L_o)^2$$

$$V_i = r_i \omega$$

$$r_i = R \frac{M_j}{r_i + r_j}$$

$$\omega^2 = \frac{(M_1 + M_2)}{M_1 M_2} \frac{k (R - L_o)}{R}$$

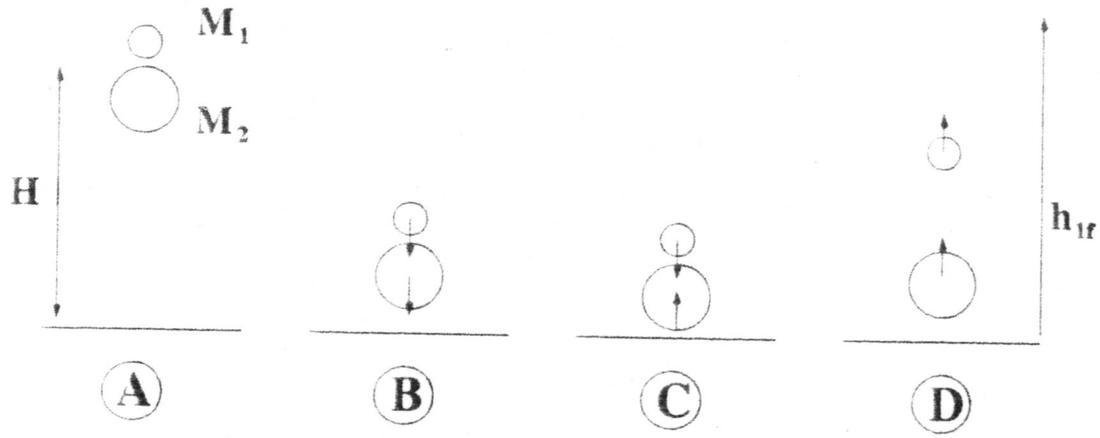
$$\boxed{E = \frac{1}{2} k (R - L_o)(2R - L_o)}$$

- 1e) (5 points) Suppose the objects started with a separation R_1 but were observed, after a long stretch of time, to be orbiting with a separation R_2. How much non-conservative work must have been done on the system during this time?

$$\Delta E = W_{NC}$$

$$W_{ext} = E(R_2) - E(R_1)$$

$$\boxed{W_{ext} = \frac{1}{2} k \left[(R_2 - L_o)(2R_2 - L_o) - (R_1 - L_o)(2R_1 - L_o) \right]}$$



2) Here's something you can try at home! Grab a large, bouncy ball (like a basketball), and a light, bouncy ball (maybe a tennis ball). Place the light ball (M_1) on top of the heavy ball (M_2) and drop them together from rest at some height (H) above the ground. Once the pair hits the ground, the lighter top ball will shoot up into the air, traveling as much as nine times higher than the height from which it was dropped.

- 2a) (5 points) How fast are M_1 and M_2 moving just before they hit the floor in figure **B**? You may assume that they are both point masses, of negligible radius.

$\Delta E = W_{NC}$

$mgh = \frac{1}{2}mv^2$

$$\boxed{V = \sqrt{2gH}}$$

- 2b) (5 points) If M_2 makes a perfectly elastic collision with the floor, how fast is it moving immediately thereafter (figure C)? Explain.

Mechanical energy is conserved - All the kinetic energy M_2 had going into the collision is returned to M_2 after the collision (in the limit where the floor doesn't move :))

$$\boxed{V_{2C} = \sqrt{2gH}}$$

- 2c) (10 points) M_2 then goes on to make a perfectly elastic collision with M_1. How fast is M_1 moving after that collision (figure **D**)?.

Elastic Collision

$$\Sigma P_{yC} = \Sigma P_{yD}$$

$$m_1(-\sqrt{2gH}) + M_2(+\sqrt{2gH}) = m_1 V_{1Dy} + M_2 V_{2Dy}$$

$$(M_2 - m_1)\sqrt{2gH} = M_1 V_{1Dy} + M_2(V_{1Dy} - 2\sqrt{2gH})$$

$$\boxed{V_{1Dy} = \frac{(3M_2 - m_1)\sqrt{2gH}}{M_1 + M_2}}$$

$$\Sigma K_C = \Sigma K_D$$

$$V_{1Cy} + V_{1Dy} = V_{2Cy} + V_{2Dy}$$

$$-\sqrt{2gH} + V_{1Dy} = +\sqrt{2gH} + V_{2Dy}$$

$$V_{2Dy} = V_{1Dy} - 2\sqrt{2gH}$$

- 2d) (5 points) A reasonable person who is rather well-versed in mechanics might take issue with the calculation you just performed. Why? Justify your work.

$\Sigma F_{ext,y} \neq 0$ (gravity acts on the system) $\Rightarrow$ $\Sigma P_y \neq$ Constant.

However ~ the Collision happens over such a brief period that the impulse delivered by gravity is negligable... we can (mostly) ignore it in this case...

- 2e) (5 points) Find the height to which M_1 will rise after the collision (h_{1f}). Show that if $M_1 << M_2$, $h_{1f} \to 9H$.

$$\frac{1}{2}mv^2 = mgh$$

$$h = \frac{v^2}{2g}$$

$$h_{1f} = \left(\frac{3m_2 - m_1}{m_1 + M_2}\right)^2 H$$

$\lim m_2 >> m_1$

$$h_{1f} \simeq 9H$$

3) Consider the collision shown above. The incident particle has a mass m_1, the target particle has a mass m_2, and the scattering angle between the particles (after the collision) is α.

- 3a) (5 points) What quantities are conserved? Why?

If m_1 and m_2 are taken together as the system, $\Sigma \vec{F}_{ext} = 0$ and momentum is conserved. "Elastic" means mechanical energy is conserved ⇒ the configuration of the system hasn't changed (immediately before/after), so kinetic energy remains constant across the collision as well...

- 3b) (5 points) Write expressions for each of the conserved quantities in terms of the (unknown) magnitudes of each particle's momentum. Use P_0 for the momentum of the incident particle, P_1 for the momentum of the incident particle after the collision and P_2 for the momentum of the target particle after the collision.

$K = P^2/2m$

$$\boxed{\begin{aligned} \vec{P_0} &= \vec{P_1} + \vec{P_2} \\ \frac{P_0^2}{2m_1} &= \frac{P_1^2}{2m_1} + \frac{P_2^2}{2m_2} \end{aligned}}$$

(1)

(2)

- 3c) (10 points) How does the magnitude of the target particle's momentum after the collision compare to the magnitude of the incident particle's momentum after the collision?

(1) $P_0^2 = P_1^2 + P_2^2 + 2\vec{P_1}\cdot\vec{P_2}$

$P_0^2 = P_1^2 + P_2^2 + 2P_1P_2\cos\alpha$

(2) $P_0^2 = P_1^2 + \frac{M_1}{M_2}P_2^2$

(1→2) $P_2^2 + 2P_1P_2\cos\alpha = \frac{M_1}{M_2}P_2^2$

$P_2\left(P_2\left(1-\frac{M_1}{M_2}\right) + 2P_1\cos\alpha\right) = 0$

$P_2 = \frac{2M_2P_1\cos\alpha}{M_1-M_2}$

$$\boxed{\frac{P_2}{P_1} = \frac{2M_2}{M_1-M_2}\cos\alpha}$$

↑ Note what happens if $M_1 = M_2$!! :)

- 3d) (10 points) What fraction of the incident particle's initial kinetic energy does it retain after the collision?

$$\frac{P_0^2}{2m_1} = \frac{P_1^2}{2m_1}\left[1 + \frac{m_1}{M_2}\left(\frac{2M_2}{m_1 - M_2}\right)^2 \cos^2\alpha\right]$$

$$\frac{P_0^2}{2m_1} = \frac{P_1^2}{2m_1}\left[1 + \frac{4m_1 M_2}{(m_1 - M_2)^2}\cos^2\alpha\right]$$

$$\boxed{\frac{K_1}{K_0} = \frac{1}{1 + \frac{4m_1 M_2}{(M_1 - M_2)^2}\cos^2\alpha}}$$