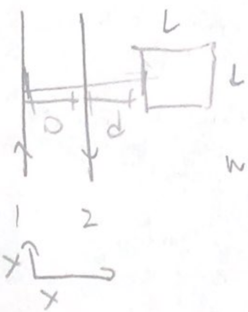
You must show your work to receive credit. An answer written down with no work will receive no credit.

(a): 10 points

Calculate the magnitude and direction of the magnetic field on the xy-plane for $x > 0$. [You may use without proof the magnetic field produced by an infinite current-carrying wire.]



$$B(r) = \frac{\mu_0 I}{2\pi r}$$

wire 1: $B(x) = -\frac{\mu_0 I}{2\pi x}\hat{z} = \frac{-\mu_0 I_0 e^{-\frac{t}{\tau}}}{2\pi x}\hat{z}$ right hand rule, points in $-\hat{z}$ direction

wire 2: $B(x) = \frac{\mu_0 I}{2\pi(x-D)} = \frac{\mu_0 I_0 e^{-\frac{t}{\tau}}}{2\pi(x-D)}\hat{z}$ right hand rule, points in $\hat{z}$ direction

Together,

for $x > 0$,

$$\boxed{B(x) = \frac{\mu_0 I_0 e^{-\frac{t}{\tau}}}{2\pi}\left(\frac{1}{(x-D)} - \frac{1}{x}\right)\hat{z}}$$

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(b): 15 points

Calculate the magnetic flux through the square bounded by the loop of wire. Take counterclockwise to be the "positive" direction for your sign convention.

$$\Phi_B = \int \vec{B} \cdot d\vec{A}$$

$$= \frac{\mu_0 I_0 e^{-\frac{t}{\tau}}}{2\pi} \int_{D+d}^{D+d+L} \left(\frac{1}{(x-D)} - \frac{1}{x} \right) dx \int_0^L dy$$

$$= \frac{\mu_0 I_0 L e^{-\frac{t}{\tau}}}{2\pi} \left(\ln(d+L) - \ln(d) - \ln(D+d+L) + \ln(D+d) \right)$$

$$= \frac{\mu_0 I_0 L e^{-\frac{t}{\tau}}}{2\pi} \left(\ln \left(\frac{(d+L)(d+D)}{d(D+d+L)} \right) \right)$$

Problem 1 continued on next page...

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(c): 15 points

Calculate the magnitude and direction of the current induced in the square loop of wire. Assume the wire is ohmic. [You may use without proof Maxwell's equations.]

$$\mathcal{E} = -\frac{d\Phi_B}{dt}$$

$$\mathcal{E} = -\frac{\mu_0 I_0 L e^{-\frac{t}{\tau}}}{2\pi} \left(\ln\left(\frac{(d+L)(d+D)}{d(d+D+L)}\right) \right) dt$$

$$\mathcal{E} = \frac{\mu_0 I_0 L e^{-\frac{t}{\tau}}}{2\pi \tau} \left(\ln\left(\frac{(d+L)(d+D)}{d(d+D+L)}\right) \right)$$

$$\mathcal{E} = IR$$

$$I = \frac{\mathcal{E}}{R} = \frac{\mu_0 I_0 L e^{-\frac{t}{\tau}}}{2\pi \tau R} \left(\ln \frac{(d+L)(d+D)}{d(d+D+L)} \right)$$

Flux decreases with time, $\vec{B}$ points out of the page, so from Lenz's Law I flows in the <u>counterclockwise direction</u>.

You must show your work to receive credit. An answer written down with no work will receive no credit.

Problem 2

30 points

Consider an LRC series circuit (resistance R, capacitance C, inductance L) being driven with an emf $V = V_0 \cos(\omega t)$, where $\omega = 1/\sqrt{LC}$. For each of the following situations (a)-(d) choose between the following responses:

1. The amplitude of the current would increase.
2. The amplitude of the current would decrease.
3. The amplitude of the current would remain the same.
4. We need more information to determine the answer.

Explain your answer in each case.

(a): 7.5 points

The capacitance of the capacitor is doubled, while holding R, L, and ω constant.

The amplitude of the current will decrease.

Increasing the capacitance will decrease the capacitive reactance X_C. However, the series circuit is being driven at its resonance frequency $\frac{1}{\sqrt{LC}}$. Increasing the capacitance will thus cause the impedance of the circuit to increase, because $Z = \sqrt{R^2 + (X_L - X_C)^2}$, and X_L no longer equals X_C. $V = IZ$, if Z increases, the current decreases.

(b): 7.5 points

The inductance of the inductor is doubled, while holding R, C, and ω constant.

The amplitude of the current will decrease.

Increasing the inductance will increase the inductive reactance, X_L. The circuit is being driven with a resonance frequence, $\frac{1}{\sqrt{LC}}$, so $X_L = X_C$. However, when X_L increases the impedance, Z, of the circuit increases, because X_L no longer equals X_C. Thus, the current decreases.

Problem 2 continued on next page...

You must show your work to receive credit. An answer written down with no work will receive no credit.

(c): 7.5 points

The resistance of the resistor is doubled, while holding L, C, and ω constant.

The amplitude of the current will decrease. The current is at its resonance frequency, so $V = IR$. Doubling the resistance will decrease the current.

(d): 7.5 points

The driving frequency is doubled, while holding R, C, and L constant.

The amplitude of the current will decrease.

If the driving frequency is doubled, while holding R, C, L constant, the circuit will no longer have its resonance frequency of $\frac{1}{\sqrt{LC}}$. This will increase the impedance of the circuit, since X_C and X_L depend directly on ω. ($X_C = \frac{1}{\omega C}$, $X_L = \omega L$). Increasing the impedance will decrease the current. (At the resonance frequency, the impedance is minimized.)

You must show your work to receive credit. An answer written down with no work will receive no credit.

Problem 3

30 points

Determine whether each of the following electric and magnetic fields could constitute a traveling electromagnetic plane wave in empty space (no charges, currents, or other matter). If yes, show that they satisfy (all four!) Maxwell's equations in vacuum (it's easier to use the differential versions; you may find the internet useful for writing down the Cartesian form of ∇). If not, explain why not.

(a): 7.5 points

$$\vec{E} = E_0 \cos(\omega(t - x/c))\,\hat{x}$$
$$\vec{B} = \frac{E_0}{c} \cos(\omega(t - x/c))\,\hat{y}.$$

No

$\boxed{\text{No}}$ Because $\nabla \cdot \vec{E} \neq 0$. The electric and magnetic fields can't have components along their propagation direction. The electric field is in a direction parallel to the wave of propagation, but it needs to be perpendicular.

(b): 7.5 points

$$\vec{E} = E_0 \cos\left(\alpha^2 x^2 + \beta^2 t^2 - 2\alpha\beta x t\right)\hat{y} \qquad (\alpha x - \beta t)^2$$
$$\vec{B} = \frac{E_0}{c} \cos\left(\alpha^2 x^2 + \beta^2 t^2 - 2\alpha\beta x t\right)\hat{z},$$

where α and β are constants such that $\beta/\alpha = c$.

Yes

$\boxed{\text{Yes.}}$ Work shown on other page.

Problem 3 continued on next page...

Problem 3(b).

$\boxed{\text{Yes}}$

$$\vec{E} = E_0 \cos(\alpha^2 x^2 + \beta^2 t^2 - 2\alpha\beta x t)\,\hat{y}$$
$$\vec{B} = \frac{E_0}{c} \cos(\alpha^2 x^2 + \beta^2 t^2 - 2\alpha\beta x t)\,\hat{z}$$

$$= \quad \vec{E} = E_0 \cos((\alpha x - \beta t)^2)\,\hat{y}$$
$$\vec{B} = \frac{E_0}{c} \cos((\alpha x - \beta t)^2)\,\hat{z}$$

$$\boxed{\nabla \times \vec{E} = -\frac{\partial B}{\partial t}}$$

$$\nabla \times \vec{E} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{d}{dx} & \frac{d}{dy} & \frac{d}{dz} \\ \underset{0}{E_x} & E_y & \underset{0}{E_z} \end{vmatrix} = E_y \frac{d}{dx}\hat{z} = \boxed{-2E_0\alpha(\alpha x - \beta t)\sin((\alpha x - \beta t)^2)}$$

$$-\frac{dB}{dt} = \frac{-2\beta E_0(\alpha x - \beta t)\sin((\alpha x - bt)^2)}{c} = \boxed{-2E_0\alpha(\alpha x - \beta t)\sin((\alpha x - \beta t)^2)}$$

$$\boxed{\nabla \times \vec{B} = \mu_0\epsilon_0 \frac{\partial \vec{E}}{\partial t}}$$

$$\nabla \times \vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{d}{dx} & \frac{d}{dy} & \frac{d}{dz} \\ \underset{0}{B_x} & \underset{0}{B_y} & B_z \end{vmatrix} = -B_z \frac{d}{dx}\hat{y} = \boxed{\frac{2E_0\alpha}{c}(\alpha x - \beta t)\sin((\alpha x - \beta t)^2)}$$

$c = \frac{1}{\sqrt{\mu_0\epsilon_0}}$

$\frac{1}{\mu_0\epsilon_0} = c^2$

$$\mu_0\epsilon_0 \frac{d\vec{E}}{dt} = \frac{2\beta}{c^2}(\alpha x - \beta t)\sin((\alpha x - \beta t)\sin((\alpha x - bt)^2) \quad \text{bc } \beta = \alpha c,$$

$$= \boxed{\frac{2E_0\alpha}{c}(\alpha x - \beta t)\sin((\alpha x - \beta t)^2)}$$

$$\boxed{\nabla \cdot \vec{E} = 0}$$

$$= \underset{0}{\frac{\partial E_x}{dx}} + \underset{0}{\frac{dE_y}{dy}} + \underset{0}{\frac{dE_z}{dz}} = 0$$

(No y component in $\vec{E}$),

$\therefore \nabla \cdot \vec{E} = 0$

$$\boxed{\nabla \cdot \vec{B} = 0}$$

$$= \underset{0}{\frac{\partial B_x}{dx}} + \underset{0}{\frac{dB_y}{dy}} + \underset{0}{\frac{dB_z}{dz}} = 0$$

(No z component in $\vec{B}$)

$\therefore \nabla \cdot \vec{B} = 0$

You must show your work to receive credit. An answer written down with no work will receive no credit.

(c): 7.5 points

$$\vec{E} = E_0 \sin(\omega(t + z/c))\,\hat{x}$$
$$\vec{B} = \frac{E_0}{c} \sin(\omega(t + z/c))\,\hat{y}.$$

$\boxed{\text{No.}}$ $\vec{\nabla} \times \vec{E} \neq -\frac{\partial \vec{B}}{\partial t}$

$$\vec{\nabla} \times \vec{E} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{d}{dx} & \frac{d}{dy} & \frac{d}{dz} \\ E_x & E_y & E_z \end{vmatrix} = E_x \frac{d}{dz}\hat{y} = \frac{\omega E_0}{c}\cos(\omega(t + \frac{z}{c}))\hat{y}$$

$E_y = 0$, $E_z = 0$

But, $-\frac{\partial B}{\partial t} = -\frac{\omega E_0}{c}\cos(\omega(t + \frac{z}{c}))\hat{y}$

(d): 7.5 points

$$\vec{E} = \frac{E_0}{\sqrt{2}} \cos(\omega(t - y/c))\,(\hat{x} + \hat{z})$$
$$\vec{B} = \frac{E_0}{c\sqrt{2}} \cos(\omega(t - y/c))\,(-\hat{x} - \hat{z}).$$

$\boxed{\text{NO.}}$

$\vec{E} \times \vec{B} = 0$; The $\vec{E}$ and $\vec{B}$ are not perpendicular.

$(\hat{x} + \hat{z}) \times (-\hat{x} - \hat{z}) = 0.$