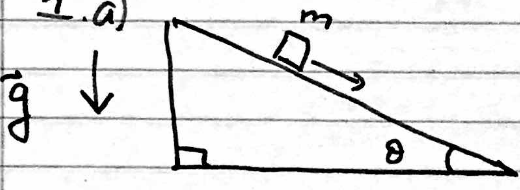
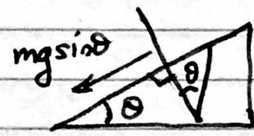
Physics 105 A

Midterm Solutions.

1. a)





$$m\frac{dv}{dt} = -kmv^2 + mg\sin\theta$$

$$\frac{dv}{dt} = -kv^2 + g\sin\theta; \quad \int_0^v \frac{dv'}{g\sin\theta - kv'^2} = t$$

$$\Rightarrow \tanh^{-1}\left[\sqrt{\frac{k}{g\sin\theta}}\, v\right]\frac{1}{\sqrt{kg\sin\theta}} = t$$

So $v(t) = \sqrt{\frac{g\sin\theta}{k}}\tanh\left[\sqrt{kg\sin\theta}\, t\right]$

Now, $\frac{d}{dx}\log(\cosh x) = \frac{\sinh x}{\cosh x} = \tanh x$ so

$$x(t) = \int_0^t v(t')\,dt' = \sqrt{\frac{g\sin\theta}{k}}\frac{1}{\sqrt{kg\sin\theta}}\ln\left[\cosh\left(\sqrt{kg\sin\theta}\, t\right)\right]$$

$\Rightarrow$ distance traveled D is given by

$$kD = \ln\left[\cosh\left(\sqrt{kg\sin\theta}\, t\right)\right] \text{ or}$$

$$e^{kD} = \cosh\left(\sqrt{kg\sin\theta}\, t\right) \Rightarrow \boxed{T = \cosh^{-1}(e^{kD})\frac{1}{\sqrt{kg\sin\theta}}}$$

b) As $D \to \infty$ the block approaches terminal velocity. ②

$$v \xrightarrow[D\to\infty]{} V^* \text{ where}$$

$$-kmV^{*2} + mg\sin\theta = 0 \Rightarrow V^* = \sqrt{\frac{g\sin\theta}{k}}$$

So the drag force is: $F_D = km\left(\frac{g\sin\theta}{k}\right) = mg\sin\theta$ in magnitude.

Power dissipated $P = Fv = F_D V^*$

$$\boxed{P = mg\sin\theta\sqrt{\frac{g\sin\theta}{k}} = \left(\frac{m^2g^3\sin^3\theta}{k}\right)^{1/2}}$$

c) The kinetic energy of the block after sliding a distance D.

$$T = \frac{1}{2}mv^2(D) \ ; \quad v(D) = \sqrt{\frac{g\sin\theta}{k}}\tanh\left[\cosh^{-1}(e^{kD})\right]$$

Using the list of formulae:

$$v(D) = \sqrt{\frac{g\sin\theta}{k}}\sqrt{\frac{e^{2kD}-1}{e^{2kD}}} = V^*\left(1-e^{-2kD}\right)^{1/2}$$

$\Rightarrow \quad T(D) = \frac{1}{2}mV^{*2}\left(1-e^{-2kD}\right) \leftarrow$ We ~~use~~ see the approach to terminal velocity.

Now, the kinetic energy without drag would have been.

$$\frac{1}{2}mv^2 = \underset{\substack{\uparrow \\ \text{drag-free}}}{T_0} = mgD\sin\theta$$

③

$$\Delta E = T_0 - T = mgD\sin\theta - \tfrac{1}{2} m V^{*2}(1 - e^{-2kD})$$

w/o drag ↑ ↑ with drag

$$\boxed{\Delta E(D) = mg\sin\theta\left\{ D - \frac{1}{2k}(1 - e^{-2kD}) \right\}}$$

Note that at first the difference ΔE grows with D as D^2

$kD \ll 1 \Rightarrow \Delta E \sim mg\sin\theta \times D^2 k$

Later on $kD \gg 1$ it grows linearly:

$kD \gg 1$; $\Delta E \sim Dmg\sin\theta$ − constant.

2.

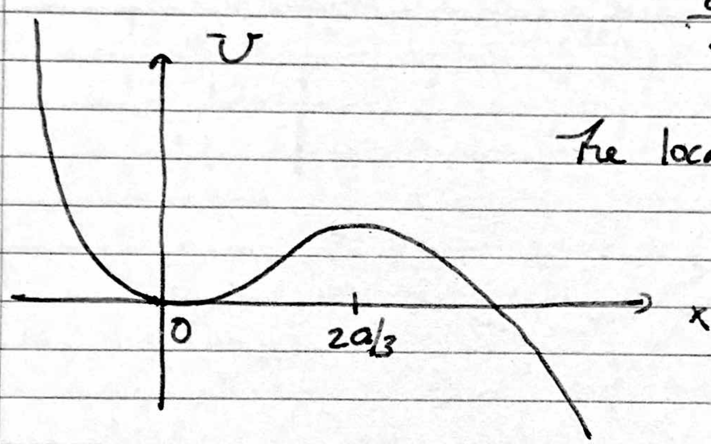
a) $U(x) = -U_0 [x^3 - ax^2]$

$$\frac{dU}{dx} = -U_0 [3x^2 - 2ax] \quad \text{zeros at } x=0 \text{ and}$$

$3x - 2a = 0 \Rightarrow x = 2a/3$.

$$\frac{d^2U}{dx^2} = -U_0 [6x - 2a]; \quad \left.\frac{d^2U}{dx^2}\right|_{x=0} = 2aU_0 > 0$$

$$\left.\frac{d^2U}{dx^2}\right|_{x=a} = -4U_0 a < 0$$



the local max is: $U_0 a^3 \cdot \frac{4}{12} = \frac{U_0 a^3}{3}$

b) Stable fixed pt at $x=0$

unstable fixed pt at $x = 2a/3$

c) expand EOM around $x=0$

$$m\ddot{x} = F(0) + x\left.\frac{dF}{dx}\right|_{x=0} = 0 - x\left.\frac{d^2U}{dx^2}\right|_{x=0}$$

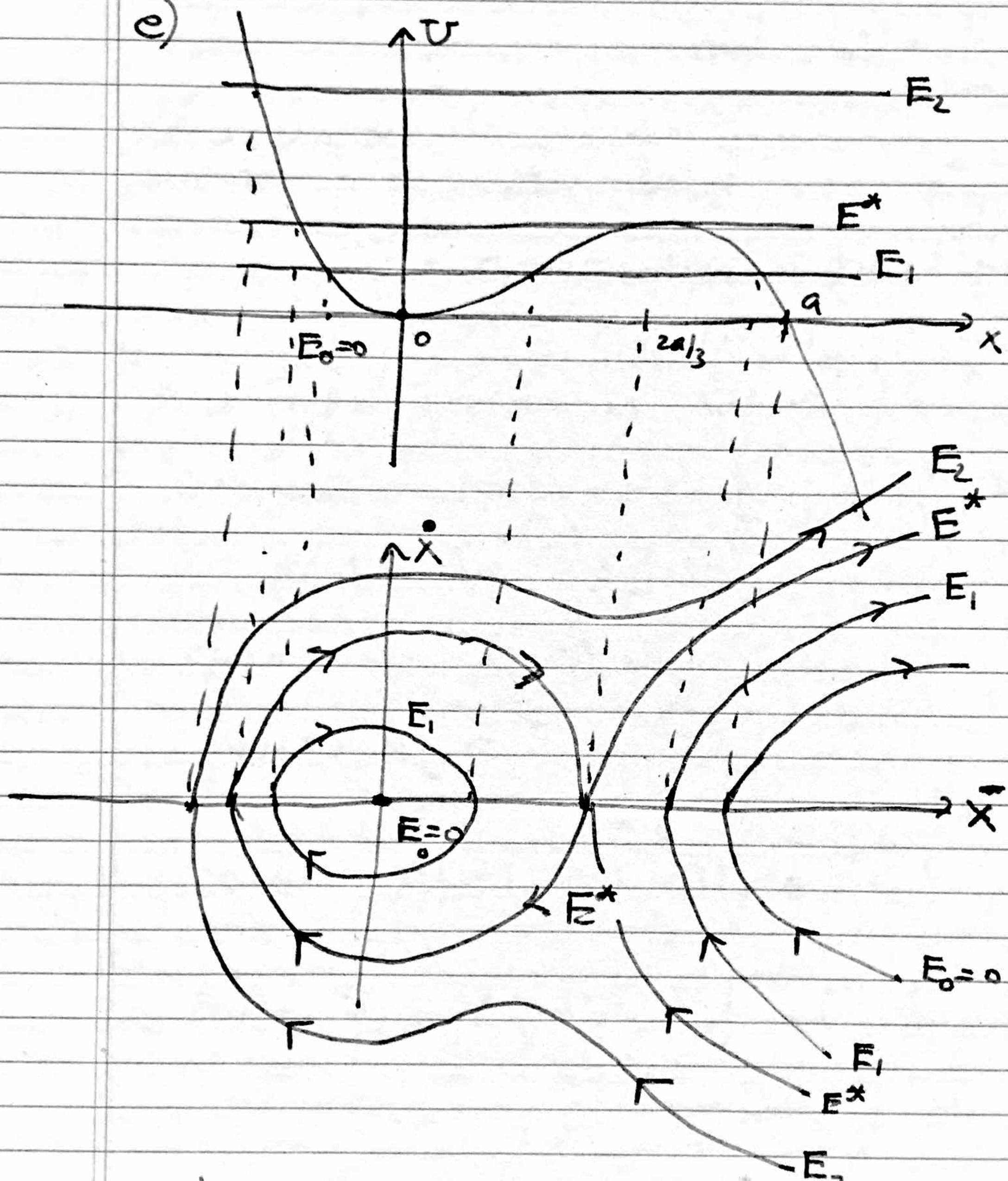
$$m\ddot{x} = -x\, 2aU_0$$

$$\Rightarrow \quad \omega_0^2 = \sqrt{\frac{2aU_0}{m}}.$$

d) For bounded oscillations we need

$$0 < E < \frac{1}{3} U_0 a^3$$

e)



Phase Space Picture.

3

a) We have oscillations obeying

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0 \quad \text{where } 2\beta = \frac{b\,2m}{2m} = b$$

$$\text{and } \omega_0^2 = \frac{k}{2m}$$

Now consider the collision:

momentum $P = MV$ before $\longleftarrow x$ take this direction as positive

$P = 2mv$ after based on the picture.

$\Rightarrow$ initial velocity $v_0 = V/2$.

and initial position is $x(0) = 0$.

We have an overdamped system.

$$x(t) = Ae^{-\Gamma_- t} + Be^{-\Gamma_+ t} \quad \text{where: } \Gamma_\pm = \beta \pm \sqrt{\beta^2 - \omega_0^2}$$

$x(0) = 0 \Rightarrow A + B = 0 \Rightarrow$

$$x(t) = A\left[e^{-\Gamma_- t} - e^{-\Gamma_+ t}\right]$$

$$\dot{x}(t) = A\left\{-\Gamma_- e^{-\Gamma_- t} + \Gamma_+ e^{-\Gamma_+ t}\right\}$$

$$\dot{x}(0) = \frac{V}{2} = A\left\{\Gamma_+ - \Gamma_-\right\} \text{ so}$$

$$\boxed{x(t) = \frac{V}{2\left[\Gamma_+ - \Gamma_-\right]}\left(e^{-\Gamma_- t} - e^{-\Gamma_+ t}\right)}$$

b) Stopping distance.

Find when $\dot{x}(t)=0$ for $t>0$.

$$0 = -\Gamma_- e^{-\Gamma_- t} + \Gamma_+ e^{-\Gamma_+ t} \quad \text{so}$$

$$\Gamma_- e^{-\Gamma_- t} = \Gamma_+ e^{-\Gamma_+ t}$$

$$\frac{\Gamma_-}{\Gamma_+} = \exp\left[-t(\Gamma_+ - \Gamma_-)\right] \quad \text{or}$$

$$t^* = -\frac{1}{\Gamma_+ - \Gamma_-}\ln\left(\Gamma_-/\Gamma_+\right)$$

← Note that this is >0 as required since $\Gamma_- < \Gamma_+$.

Now plug back into the EOM

$$x(t^*) = \frac{V}{2[\Gamma_+ - \Gamma_-]}\left(e^{-\Gamma_- t^*} - e^{-\Gamma_+ t^*}\right) \quad \text{or}$$

$$x(t^*) = \frac{V e^{-\Gamma_- t^*}}{2[\Gamma_+ - \Gamma_-]}\left(1 - \frac{\Gamma_-}{\Gamma_+}\right) = \frac{V}{2\Gamma_+} e^{-\Gamma_- t^*}$$

Finally, plug in t^* from above:

$$-\Gamma_- t^* = \frac{\Gamma_-}{\Gamma_+ - \Gamma_-}\ln\left(\Gamma_-/\Gamma_+\right)$$

$$e^{-\Gamma_- t^*} = \left(\frac{\Gamma_-}{\Gamma_+}\right)^{\frac{\Gamma_-}{\Gamma_+ - \Gamma_-}} \quad \text{and}$$

$$\boxed{D = x(t^*) = \frac{V}{2\Gamma_+}\left(\frac{\Gamma_-}{\Gamma_+}\right)^{\Gamma_-/(\Gamma_+ - \Gamma_-)}}$$

4. Using the hint, we can write:

$$x(t) = \int dt' \, \frac{F(t')}{m\omega_1} e^{-\beta(t-t')} \sin[\omega_1(t-t')]$$

taking $\beta \to 0$ and using the fact that the force is on for a period $T = 2\pi/\omega_0$

$$x(t) = \int_{-T/2}^{T/2} dt' \, \frac{F_0}{m\omega_0} \sin(\omega_0 t') \sin[\omega_0(t-t')]$$

expand the sine:

$$x(t) = \frac{F_0}{m\omega_0} \int_{-T/2}^{T/2} dt' \sin(\omega_0 t') \{ \sin(\omega_0 t)\cos(\omega_0 t') - \cos(\omega_0 t)\sin(\omega_0 t') \}$$

$$x(t) = 0 - \frac{F_0}{m\omega_0} \cos(\omega_0 t) \int_{-T/2}^{+T/2} dt' \sin^2(\omega_0 t') = -\frac{F_0 \cos(\omega_0 t)}{m\omega_0} \frac{T}{2}$$

Note that the amplitude will grow linearly with time of force (application $\Rightarrow$ resonance with no damping makes x blow up eventually.

$\Rightarrow$ for $t > T/2$ we have

$$\boxed{x(t) = -\frac{F_0}{m\omega_0} \left(\frac{\pi}{\omega_0} \right) \cos(\omega_0 t)}$$