MECHANICAL & AEROSPACE ENGINEERING DEPARTMENT
UNIVERSITY OF CALIFORNIA, LOS ANGELES

MAE 82

MATHEMATICS OF ENGINEERING

WINTER, 2019

INSTRUCTOR

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MIDTERM EXAMINATION
Closed Book and Closed Notes

February 14, 2019
12-1:45pm

INSTRUCTIONS: SHOW ALL CALCULATIONS ON THESE PAGES. ATTACH ADDITIONAL PAGES AS NECESSARY.

	Maximum Score	Your Score
Problem 1	20	20
Problem 2	20	18
Problem 3	20	19
Total	60	57

NAME [redacted]

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Problem 1 (20)

The displacement $u(t)$ of the mass of a dynamic (m,c,k) system, is governed by the following second order, linear, differential equation (SLDE) $m\ddot{u}+c\dot{u}+ku=F(t)$, $u(0)=u_0$, $\dot{u}(0)=\dot{u}_0$ where m is the mass, c is the damping coefficient, k is the linear spring constant, and $F(t)$ is the external applied force on the mass.

(a) Modify the above governing differential equation for the following cases. 1) Free, undamped motion **(2)**, 2) Free, damped motion **(2)**, 3) Free critically damped motion **(2)**, 4) Free underdamped motion **(2)**, 5) Forced underdamped motion with Harmonic Loading **(2)**

$\zeta = \frac{c}{2m\omega_n}$

20

(b) Consider the vibration of an undamped dynamic system under an external harmonic loading $F(t)=F_0\cos(\omega t), F_0=\text{constant}$. **Derive** the complete solution of $u(t)$, $u(0)=0$, $\dot{u}(0)=0$ **(6)**, and (c) discuss the characteristic difference between the phenomenon of **Beats and Resonance** **(4)**

(a) 10

a) $m\ddot{u}+c\dot{u}+ku=F(t)$ $u(0)=u_0$, $\dot{u}(0)=\dot{u}_0$

1) Free, undamped motion; $F(t)=0$, $c=0$ $\boxed{m\ddot{u}+ku=0}$ ✓

2) Free, damped motion: $F(t)=0$, $c\neq 0$: $\boxed{m\ddot{u}+c\dot{u}+ku=0}$ ✓

3) Free critically damped motion ; $F(t)=0$, $c=2\sqrt{km}$, $\zeta=1=\frac{c}{2m\omega_n}$ $\boxed{m\ddot{u}+c\dot{u}+ku=0}$ ✓

4) Free underdamped motion : $\zeta<1$ $c<2\sqrt{km}$ $F(t)=0$ $\boxed{m\ddot{u}+c\dot{u}+ku=0}$ ✓

5) Forced underdamped motion w/ Harmonic loading : $F(t)=F_0e^{i\omega t}$ $c<2\sqrt{km}$ $\zeta<1$

$\boxed{m\ddot{u}+c\dot{u}+ku=F_0e^{i\omega t}}$ ✓

b) undamped $c=0$

$m\ddot{u}+ku=F_0\cos(\omega t)$

$y=e^{\alpha t}$

$y'=\alpha e^{\alpha t}$

$y''=\alpha^2 e^{\alpha t}$

Solve for fundamentary

$(m\alpha^2+k)e^{\alpha t}=0$

$m\alpha^2+k=0$

$\alpha^2=\frac{-k}{m}=$

$\alpha=\sqrt{\frac{-k}{m}}=i\omega_n$ ✓

(Next page)

$\alpha = \omega_n i \quad y = e^{\omega_n i t}$

$y_c = C_1\cos(\omega_n t) + C_2\sin(\omega_n t)$ ✓

$g(t) = F_o\cos(\omega t)$

$y_p = A\cos(\omega t) + B\sin(\omega t)$

$y_p' = -A\omega\sin(\omega t) + B\omega\cos(\omega t)$

$y_p'' = -A\omega^2\cos(\omega t) - B\omega^2\sin(\omega t)$

$\frac{F_o}{K - m\omega^2} \to \frac{F_o}{\frac{mK}{m} - m\omega^2} = \frac{F_o}{m(\omega_n^2 - \omega^2)}$

Plug into original equation

$-Am\omega^2\cos(\omega t) - Bm\omega^2\sin(\omega t) + AK\cos(\omega t) + BK\sin(\omega t) = F_o\cos(\omega t)$

set coefficients equal to each other

$\cos(\omega t)$: $-Am\omega^2 + AK = F_o$

$A(K - m\omega^2) = F_o$

$A = \frac{F_o}{K - m\omega^2} = \frac{F_o}{m(\omega_n^2 - \omega^2)}$

$\sin(\omega t)$: $-Bm\omega^2 + BK = 0$

$B(K - m\omega^2) = 0$

$B = 0$

$y_p = \frac{F_o\cos(\omega t)}{m(\omega_n^2 - \omega^2)}$ ✓

$y(t) = C_1\cos(\omega_n t) + C_2\sin(\omega_n t) + \frac{F_o\cos(\omega t)}{m(\omega_n^2 - \omega^2)} = 0$

$y'(t) = -C_1\omega_n\sin(\omega_n t) + C_2\omega_n\cos(\omega_n t) - \frac{F_o\omega\sin(\omega t)}{m(\omega_n^2 - \omega^2)} = 0$

plug in $u(0) = 0$ & $\dot{u}(0) = 0$

$C_1 + \frac{F_o}{m(\omega_n^2 - \omega^2)} = 0$

$C_1 = \frac{-F_o}{m(\omega_n^2 - \omega^2)}$

$C_1\cancel{(0)} + C_2\omega_n - \cancel{\beta} = 0$

$\boxed{C_2 = 0}$

b) 6

$y(t) = \frac{-F_o}{m(\omega_n^2 - \omega^2)}\cos(\omega_n t) + \frac{F_o\cos(\omega t)}{m(\omega_n^2 - \omega^2)}$

$\boxed{y(t) = \frac{F_o}{m(\omega_n^2 - \omega^2)}\left(\cos(\omega t) - \cos(\omega_n t)\right)}$ ✓

$2\beta = \omega t - \omega_n t$

$\alpha = \frac{\omega_n t + \omega t}{2}$

$\alpha + \beta = \omega t$

$\alpha - \beta = \omega_n t$

$2\beta = \omega t - \omega_n t$

Using difference of cosines

$\cos(\omega t) = \cos\alpha\cos\alpha - \sin\alpha\sin\beta$

$\cos(\omega_n t) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

$\cos(\omega t) - \cos(\omega_n t) = -2\sin\alpha\sin\beta$

$\alpha = \frac{\omega_n t + \omega t}{2}$

$\beta = \frac{\omega t - \omega_n t}{2}$

$y(t) = \frac{2F_o}{m(\omega_n^2 - \omega^2)}\left(\sin\left(\frac{\omega_n t + \omega t}{2}\right)\sin\left(\frac{\omega t - \omega_n t}{2}\right)\right)$

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Problem 2 (20)

Consider the nonhomogeneous third order DE $y^{(3)} - 2y'' - 21y' - 18y = 3 + 4e^{-t}$

(a) Determine the fundamental solutions of the homogeneous DE **(8)**

(b) Derive the particular solution using the method of variation of parameters **(12)**

Hint: Use the identity $x^3 - 2x^2 - 21x - 18 \equiv (x+1)(x+3)(x-6)$

You may use this variation of parameters formula, $Y(t) = \sum_{m=1}^{n} y_m(t) \int^t \frac{g(s)W_m(s)}{W(s)} ds$

$y^{(3)} - 2y'' - 21y' - 18y = 0$ $\quad y = e^{\alpha t}$, $y' = \alpha e^{\alpha t}$, $y'' = \alpha^2 e^{\alpha t}$, $y''' = \alpha^3 e^{\alpha t}$

converting y to be $e^{\alpha t}$ and plugging into homogenous equation. Factoring and solving for α

a) $(\alpha^3 - 2\alpha^2 - 21\alpha - 18)e^{\alpha t} = 0$

$(\alpha+1)(\alpha+3)(\alpha-6) = 0$

$\alpha = -1, -3, 6$

$$\boxed{y_c = c_1 e^{-t} + c_2 e^{-3t} + c_3 e^{6t}}$$

$$W = \begin{vmatrix} e^{-t} & e^{-3t} & e^{6t} \\ -e^{-t} & -3e^{-3t} & 6e^{6t} \\ e^{-t} & 9e^{-3t} & 36e^{6t} \end{vmatrix} \begin{matrix} e^{-t} & e^{-3t} \\ -e^{-t} & -3e^{-3t} \\ e^{-t} & 9e^{-3t} \end{matrix}$$

$W = -108e^{2t} + 6e^{2t} - 9e^{2t} + 3e^{2t} - 54e^{2t} + 36e^{2t}$

$W = -126e^{2t}$

b) $$W_1(t) = \begin{vmatrix} 0 & e^{-3t} & e^{6t} \\ 0 & -3e^{-3t} & 6e^{6t} \\ 1 & 9e^{-3t} & 36e^{6t} \end{vmatrix} \begin{matrix} 0 & e^{-3t} \\ 0 & -3e^{-3t} \\ 1 & 9e^{-3t} \end{matrix}$$

$6e^{3t} + 3e^{3t} = 9e^{3t}$

$$W_2(t) = \begin{vmatrix} e^{-t} & 0 & e^{6t} \\ -e^{-t} & 0 & 6e^{6t} \\ e^{-t} & 1 & 36e^{6t} \end{vmatrix} \begin{matrix} e^{-t} & 0 \\ -e^{-t} & 0 \\ e^{-t} & 1 \end{matrix}$$

$-e^{5t} - 6e^{5t} = -7e^{5t}$

$$W_3(t) = \begin{vmatrix} e^{-t} & e^{-3t} & 0 \\ -e^{-t} & -3e^{-3t} & 0 \\ e^{-t} & 9e^{-3t} & 1 \end{vmatrix} \begin{matrix} e^{-t} & e^{-3t} \\ -e^{-t} & -3e^{-3t} \\ e^{-t} & 9e^{-3t} \end{matrix} = -3e^{-4t} + e^{-4t} = -2e^{-4t}$$

(Continued on Back)

$g(s) = 3 + 4e^{-s}$

$$\boxed{\begin{aligned} Y(t) &= e^{-t}\int^{t} \frac{(3+4e^{-s})\,9e^{3s}}{-126e^{2s}}\,ds + e^{-3t}\int^{t}\frac{(3+4e^{-s})-7e^{5s}}{-126e^{2s}}\,ds \\ &+ e^{6t}\int^{t}\frac{(3+4e^{-s})-2e^{-4s}}{-126e^{2s}}\,ds \end{aligned}}$$

used $Y(t) = \sum_{m=1}^{n} Y_m(t)\int^{t}\frac{g(s)\,W_m(s)}{W(s)}\,ds$

$$Y_1(t) = e^{-t}\int \frac{27e^{3s} + 36e^{2s}}{-126e^{2s}} = \frac{9}{-42} = \int^{t} \frac{-3}{14}e^{s} - \frac{3}{7}\,ds$$

$$= e^{-t}\left(\frac{-3}{14}e^{t} - \frac{3}{7}t\right)$$

$$Y_2(t) = e^{-3t}\left(\int^{t}\frac{-21e^{5s}}{-126e^{2s}} - \frac{28e^{4s}}{-126e^{2s}}\,ds\right)$$

$$e^{-3t}\left(\int^{t}\frac{1}{6}e^{3s} + \frac{2}{9}e^{2s}\,ds\right) = e^{-3t}\left(\frac{1}{18}e^{3t} + \frac{1}{9}e^{2t}\right)$$

$$Y_3(t) = e^{6t}\int^{t}\frac{-6e^{-4s} - 8e^{-5s}}{-126e^{2s}}\,ds = e^{6t}\int^{t}\left(\frac{1}{21}e^{-6s} + \frac{4}{63}e^{-7s}\right)ds$$

$$= e^{6t}\left(\frac{-1}{126}e^{-6t} - \frac{4}{441}e^{-7t}\right)$$

$$\cancel{Y_p(t) = e^{-t}\left(\frac{-3}{14}e^{t} - \frac{3}{7}t\right) + e^{-3t}\left(\frac{1}{18}e^{3t} + \frac{1}{9}e^{2t}\right) + e^{6t}\left(\frac{-1}{126}e^{-6t}\right.}$$

$$\boxed{Y_p(t) = e^{-t}\left(\frac{-3}{14}e^{t} - \frac{2}{7}t\right) + e^{-3t}\left(\frac{1}{18}e^{3t} + \frac{1}{9}e^{2t}\right) + e^{6t}\left(\frac{-1}{126}e^{-6t} - \frac{4}{441}e^{-7t}\right)}$$

X

-2

Problem 3 (20)

(a) Consider the power series solution of the SLDE $(2+x^2)y''-xy'+4y=0$ about $x=0$
Derive the recurrence relation between the coefficients of the PS solution **(10)**
(b) Determine the first three non-zero terms of the power series expansions of the fundamental solutions. **(6)**
(c) What is the radius of convergence ρ of the power series solution? Explain your method of solving for ρ **(4)**

a) $P(x) = 2+x^2$

$P(x_0) = P(0) = 2 \neq 0$ Because $P(x_0) \neq 0$, the point is an ordinary point and we can treat $y = \sum_0^\infty a_n (x-x_0)^n$

$x_0 = 0$

$y' = \sum_1^\infty n a_n (x-x_0)^{n-1}$

$y = \sum_{-\infty}^{\infty} a_n (x-x_0)^n$ for $a_n = 0$ when $n < 0$ $\quad y'' = \sum_2^\infty n(n-1)a_n (x-x_0)^{n-2}$

$$(2+x^2)(n(n-1)a_n x^{n-2}) - x n a_n x^{n-1} + 4a_n x^n = 0$$

$$2n(n-1)a_n x^{n-2} + n(n-1)a_n x^n - n a_n x^n + 4 a_n x^n = 0$$

need to get to x^n

$$\underbrace{(2(n+2)(n+1)a_{n+2} + n(n-1)a_n - n a_n + 4a_n)\, x^n = 0}_{\text{recurrence relation}}$$

for all x in $|x| < \rho$

-1

$$\boxed{a_{n+2} = \frac{n a_n - 4a_n - n(n-1)a_n}{2(n+2)(n+1)}}$$

b) $a_2 = \frac{-4a_0}{4} = -a_0$; $a_3 = \frac{a_1 - 4a_1}{12} = \frac{-3a_1}{12} = \frac{-a_1}{4}$; $a_4 = \frac{2a_2 - 4a_2 - 2a_2}{24} = \frac{-a_2}{6} = \frac{a_0}{6}$

plug in a_2

$a_5 = \frac{3a_3 - 4a_3 - 3(2)a_3}{40} = \frac{3a_3 - 4a_3 - 6a_3}{40} = \frac{-7a_3}{40} = \frac{7a_1}{160}$;

$y(t) = a_0 Y_1 t + a_1 Y_2(t)$

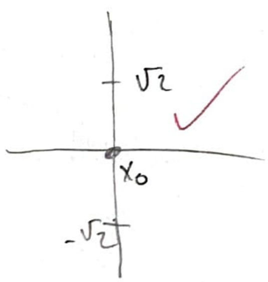
✗. plug in a_n and X^n for the series

$$\boxed{\begin{aligned} Y_1(t) &= 1 - x^2 + \frac{x^4}{6} \\ Y_2(t) &= X - \frac{X^3}{4} + \frac{7x^5}{160} \end{aligned}}$$

c) The radius of convergence of a power series is the distance away from x_0 that the power series will apply. It is the "neighborhood" of points. You can determine this by finding the singular points of the function by setting $P(x) = 0$ and seeing how far away x_0 is from a singular point

$P(X) = 2 + x^2 = 0$ $\quad x^2 = -2$

$X = \pm\sqrt{2}\,i$



Thus, the distance to a singular point is $\sqrt{2}$, so the radius of convergence

$$\boxed{\rho = \sqrt{2} \text{ at } X_0 = 0}$$