Problem 1)

Let A be the event where an ace is drawn from first deck
Let B be the event where the card chosen from second deck is the one that was in first deck
Let C be the event where an ace is drawn from second deck

$$P(C|A) = P_A(C)$$

$$P_A(C) = P_A(C|B) \cdot P_A(B) + P_A(C|B^c) P_A(B^c)$$

$$P_A(B) = P(B|A) = \frac{1}{27} \qquad P_A(B^c) = P(B^c|A) = \frac{26}{27}$$

$$P_A(C|B) = 1$$

Let A_i be the event where the second deck originally had i aces

$$P_A(C|B^c) = P_A(C|B^c \cap A_0) \cdot P_A(A_0) + P_A(C|B^c \cap A_1) \cdot P_A(A_1)$$
$$+ P_A(C|B^c \cap A_2) P_A(A_2) + P_A(C|B^c \cap A_3) P_A(A_3)$$
$$+ P_A(C|B^c A_4) P_A(A_4)$$

B is independent from A_i

$$P(A \cap A_0) = P(A|A_0) \cdot P(A_0) = \frac{4}{26} \cdot \frac{1}{2^4}$$

$$P(A \cap A_1) = P(A|A_1) \cdot P(A_1) = \frac{3}{26} \cdot \frac{4}{2^4}$$

$$P(A \cap A_2) = P(A|A_2) \cdot P(A_2) = \frac{2}{26} \cdot \frac{{}_4C_2}{2^4}$$

$$P(A \cap A_3) = \frac{1}{26} \cdot \frac{4}{2^4}$$

$$P(A \cap A_4) = P(A|A_4) \cdot P(A_4) = \frac{0}{26} \cdot \frac{1}{2^4} = 0$$

$$P(A) = \frac{1}{13}$$

$$P_A(A_0) = \frac{P(A \cap A_0)}{P(A)} = \frac{\frac{4}{26} \cdot \frac{1}{2^4}}{\frac{1}{13}} = \frac{1}{8} \qquad P_A(A_1) = \frac{P(A \cap A_1)}{P(A)} = \frac{\frac{3}{26} \cdot \frac{4}{2^4}}{\frac{1}{13}} = \frac{3}{8}$$

$$P_A(A_2) = \frac{P(A \cap A_2)}{P(A)} = \frac{3}{8} \qquad P_A(A_3) = \frac{P(A \cap A_3)}{P(A)} = \frac{1}{8} \qquad P_A(A_4) = 0$$

Problem 1 cont.)

$$P_A(C|B^c) = \frac{0}{26}\cdot\frac{1}{8} + \frac{1}{26}\cdot\frac{3}{8} + \frac{2}{26}\cdot\frac{3}{8} + \frac{3}{26}\cdot\frac{1}{8} + \frac{4}{26}\cdot\frac{0}{8}$$
$$= \frac{3}{52}$$

$$P_A(C) = P_A(C|B)\cdot P(B) + P_A(C|B^c)\cdot P(B^c)$$
$$= \frac{1}{27} + \frac{3}{52}\cdot\frac{26}{27} = \boxed{\frac{5}{54}}$$

Problem 2)

a. P_n is composed of 2 disjoint cases: the n^{th} trial is a success and an odd number of successes happened in $n-1$ trials, and the n^{th} trial is a failure and an even number of successes happened in $n-1$ trials.
The former's probability can be calculated as probability of successful trial multiplied by probability of odd # of successes in $n-1$ trials, which equals $p(1-P_{n-1})$

The latter's probability can be calculated as probability of unsuccessful trial multiplied by probability of even # of successes in $n-1$ trials, which equals $(1-p)(P_{n-1})$

We can sum the two cases as they are disjoint, which

gives us $P_n = p(1-P_{n-1}) + (1-p)P_{n-1}$; $n \geq 1$ since

we can assume 0 is an even number and $P_0 = 1$.

b. $n=1$:
$$P_1 = \frac{1+(1-2p)}{2} = 1-p = p(1-P_0)+(1-p)P_0 = 1-p \checkmark$$

Assume for inductive step that this holds for $n-1$:

$$P_n = p(1-P_{n-1}) + (1-p)P_{n-1}$$
$$= \left(1-\tfrac{1}{2}\left(1+(1-2p)^{n-1}\right)\right)p + \tfrac{1}{2}\left(1+(1-2p)^{n-1}\right)(1-p)$$
$$= p - \tfrac{1}{2}p - \tfrac{1}{2}p(1-2p)^{n-1} + \tfrac{1}{2}(1-p) + \tfrac{1}{2}(1-p)(1-2p)^{n-1}$$
$$= p - p + \tfrac{1}{2} + \tfrac{1}{2}(1-2p)(1-2p)^{n-1} = \tfrac{1}{2}\left(1+(1-2p)^n\right)$$

Since we have shown base case is true & we proved P_n is true with P_{n-1}, we have finished the inductive proof.

Problem 3) Let X be a random variable for how many defective items in the Sample

X can be $0, 1, 2, 3$

Let $0 \le k \le 3$

$$P(X=k) = \frac{{}_4C_k \cdot {}_{16}C_{3-k}}{{}_{20}C_3}$$

$$E(X) = \sum_{k=0}^{3} k \cdot P(X=k) = \sum_{k=0}^{3} k \cdot \frac{{}_4C_k \cdot {}_{16}C_{3-k}}{{}_{20}C_3}$$

$$= \frac{1}{{}_{20}C_3}\left({}_4C_1 \cdot {}_{16}C_2 + 2 \cdot {}_4C_2 \cdot {}_{16}C_1 + 3 \cdot {}_4C_3 \cdot {}_{16}C_0\right)$$

$$= \frac{684}{1140} = 0.6$$

0.6 expected number of defective items in the sample

Problem 4)

Possible permutations (no duplicates) $= \frac{11!}{5! \cdot 2! \cdot 2!} = 83160$

83160 unique permutations