Math 61: Introduction to Discrete Structures
Midterm #2

Instructor: Spencer Unger

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Name: Solutions

ID # ______

Section ______

Good Luck! Be sure to justify your answers!
No calculators, books or notes are allowed.

Problem	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
Total	100	

Version 2

1. Do the following:

(a) (10 points) Solve the following recurrence relation:

$$a_n = 6a_{n-1} - 9a_{n-2}$$

with the initial conditions $a_0 = 2$ and $a_1 = 21$.

$$r^2 = 6r - 9$$

$$0 = r^2 - 6r + 9$$

$$= (r-3)^2$$

So $a_n = C3^n + Dn3^n$

$2 = a_0 = C + D(0)$

$21 = a_1 = 3C + 3(1)D$

$\Big\}$ $C = 2$, $D = 5$

Finally $a_n = 2 \cdot 3^n + 5n3^n$

Problem 1 is continued on the next page.

(b) (10 points) Let d_n be the number of strings of zeros and ones which do not contain 111 as a substring. Find a recurrence relation that d_n satisfies. Don't forget the initial conditions and be sure to explain which counting principles you used to obtain your answer.

Essentially the same as version 1.

Very similar to version 1.

2. (20 points) Do the following:

   (a) Imagine a trial in which you flip a coin 31 times in a row. Order matters in the outcome of the trial.

      i. Count the number of outcomes which have an odd number of heads.

      ii. Count the number of outcomes which have an even number of heads.

      iii. Find a simple formula for the sum of your answers from (a) and (b). Justify your answer.

Problem 2 is continued on the next page.

Very similar to version 1.

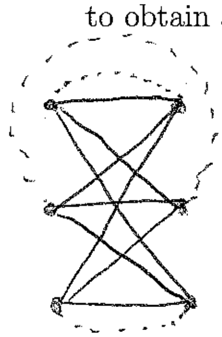
(b) Show that if you choose 15 numbers between 1 and 25 there must be two which differ by 4.

3. (20 points) Do the following:

   (a) Consider the graph $K_{3,3}$. Answer the following questions:

      i. What is the largest number of edges you can remove and still have a Hamilton cycle? Draw such a graph. 3 edges

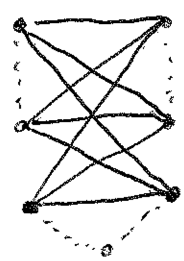
      ii. What is the smallest number of edges you need to add (multiple edges allowed) to obtain a graph with an Euler cycle? Draw such a graph.



3 edges

Added edges are dashed.

      iii. What are the smallest numbers of edges and vertices you need to add to create a simple graph (no multiple edges allowed) with an Euler cycle? Draw such a graph. 1 vertex and 4 edges



   (b) A path is simple if it does not visit the same vertex twice. Let k be a natural number. Count the number of simple paths in K_{45} of length k.

$$\frac{45!}{(45-(k+1))!} \qquad \cancel{\frac{45!}{(45-k)!}}$$

Since all edges are present the number of simple paths is the number of k+1-arrangements from 45.

Same as version 1

4. (20 points) For the purpose of this question a graph is a simple graph, that is *no* self-loops and *no* multiple edges. A graph G is n-connected if it remains connected after removing any $n-1$ edges. Do the following:

   (a) Draw a graph which is 1-connected, but not 2-connected.

   (b) Draw a graph which is 2-connected, but not 3-connected

   (c) Prove that if $n < k$ and a graph G is k-connected, then G is n-connected.

Same as version 1.

5. (20 points) Consider the graph $G = (V, E)$ where $V = \mathbb{R} \times \mathbb{R}$ and $\{x, y\} \in E$ if and only if the distance from x to y is 1. Answer the following questions about G.

   (a) Answer true or false for each of the following:

      i. $\{(0,0), (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})\} \in E$

      ii. $\{(1,0), (0,1)\} \in E$

      iii. There is a path of length 7 from $(0,0)$ to $(5,5)$.

      iv. There is a path of length 8 from $(0,0)$ to $(5,5)$.

      v. For any vertices p_1, p_2 and p_3, if $\{p_1, p_2\} \in E$ and $\{p_2, p_3\} \in E$, then $\{p_1, p_3\} \in E$.

   (b) Given a point $p \in V$, describe the set $\{x \in V \mid \{x, p\} \in E\}$. Drawing a picture is fine.

   (c) Prove or disprove: G is connected.