Version 1

MATH 61: INTRODUCTION TO DISCRETE STRUCTURES
MIDTERM #1

INSTRUCTOR: SPENCER UNGER

Name: Solutions
ID # ____________________
Section __________
Good Luck! Be sure to justify your answers!
No calculators, books or notes are allowed.

Problem	Points	Score
1	15	
2	15	
3	15	
4	15	
5	15	
Total	75	

Be careful, there are problems on both sides of the paper!

Date: April 18, 2016.

(1) (15 points) Prove by induction that for all $n \geq 4$, $2^n \geq n^2$.

Base Case $n=4$

$2^4 = 16$ and $4^2 = 16$

So $2^4 \geq 4^2$

Inductive step: Assume that $2^n \geq n^2$ for some $n \geq 4$.

$$(n+1)^2 = n^2 + 2n + 1 \leq n^2 + n^2 \leq 2^n + 2^n = 2 \cdot 2^n = 2^{n+1}$$

this uses the induction hypothesis.

here we are using that $n^2 \geq 2n+1$ when $n \geq 4$.

(3) (15 points; each of (a), (b) and (c) are weighted equally) Answer the following questions.

(a) Let $A = \{a, b, c\}$, $B = \{1, 2, 3\}$ and $C = \{b, c, 2, 3\}$. Determine each of the following. If it is a set, then write the set by listing the elements. If it is a number, then write which one. If it is a statement, then write true or false.

(i) $|B \cup C|$

5

(ii) $A \times B$

$\{(a,1),(a,2),(a,3),(b,1),(b,2),(b,3),(c,1),(c,2),(c,3)\}$

(iii) $(A \cup B) - C$

$\{a,1\}$

(iv) $C \cap (B \times B)$

$\emptyset$

(v) $(A \cap B) \subseteq C$

True

(b) Let A_1 and A_2 be sets. Find *disjoint* sets B_1 and B_2 such that $A_1 \cup A_2 = B_1 \cup B_2$.

Set $B_1 = A_1 - A_2$

$B_2 = A_2$

Now $A_1 \cup A_2 = B_1 \cup B_2$ and $B_1 \cap B_2 = \emptyset$

(c) Prove or disprove the following statement. For any sets A, B and X, if $A \subseteq B \subseteq X$, then $X - A \subseteq X - B$.

This is false. Take $A = \{a\}$ $B = \{a,b\}$ $X = \{a,b,c\}$

Clearly $A \subseteq B \subseteq X$, but

$X - A = \{b,c\} \not\subseteq \{c\} = X - B$.

$\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

(2) (15 points) Consider the set A of points on the circle of radius 1 centered at the origin. Define a relation R on A by $(p, q) \in R$ exactly when q can be obtained by moving p some multiple of ~~45 degrees~~ around the circle *in either direction!*. So for example $(1, 0)$ is related to $(-\sqrt{2}, \sqrt{2})$ since we can move from the first point to the second by moving counter~~clockwise by~~ 45 degrees *three times*. However, $(0, 1)$ is not related to $(.5, \sqrt{1 - .5^2})$, because no amount of rotating by 45 degrees will move us between the points.

(a) Show that R is an equivalence relation.

Reflexive   For any point $p \in A$ $(p, p) \in R$ since we can get from p to itself by moving 45 degrees zero times.

Symmetric   For any points $p, q \in A$, assume $(p, q) \in R$. So there is a whole number k so that moving 45° k times ~~goes~~ in one direction goes from p to q. To see that $(q, p) \in A$ move 45° k times in the opposite direction.

Transitive   For any points $p, q, r \in A$ if $(p, q) \in R$ and $(q, r) \in R$ then we get a multiple of 45° for each say k_1 and k_2.
To see $(p, r) \in R$ move p either $(k_1 + k_2) \times 45^\circ$ or $|k_1 - k_2| \times 45^\circ$ either clockwise or counter clockwise based on the direction between p and q and q and r.

(b) List the elements of $[(1, 0)]$.

$(1, 0)$, $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$, $(0, 1)$, $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$, $(-1, 0)$, $\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$, $(0, -1)$, $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$

(4) (15 points) Define the following sets:

(a) Let X be the set of strings with alphabet $\{0, 1\}$.

(b) Let Y be the set of strings with alphabet $\{a, b\}$.

(c) Let Z be the set of strings with alphabet $\{0, 1, a, b\}$.

Define a function $F : X \times Y \to Z$ by $F(\alpha, \beta) = \alpha \frown \beta$. Recall that $\alpha \frown \beta$ is the concatenation of α and β.

(a) Is F injective? Justify your answer. Yes

If $F(\alpha, \beta) = F(\gamma, \delta)$ then $\alpha \frown \beta = \gamma \frown \delta$

So the length of α must be the length of γ and similarly for β and δ, since the alphabets don't overlap.

Now we get $\alpha = \gamma$ and $\beta = \delta$ since the corresponding entries must be equal.

(b) Is F surjective? Justify your answer.

No. The string a1 is not in the range since there is an a to the left of a 1.

So it can't be in the form $\alpha \frown \beta$ where $\alpha \in X$ and $\beta \in Y$.

(c) Recall that the range of F is the set $\{z \in Z \mid$ there is $(x, y) \in X \times Y$ such that $F(x, y) = z\}$. Show that $X \cup Y$ is a subset of the range of F.

Let λ denote the empty string

Now if $\alpha \in X$ $F(\alpha, \lambda) = \alpha \frown \lambda = \alpha$

if $\beta \in Y$ $F(\lambda, \beta) = \lambda \frown \beta = \beta$

So $X \cup Y$ is contained in the range of F.

(5) (15 points) Consider the numbers between 10000 and 99999 inclusive. **To receive full credit you must explain which counting principles you use and how you are using them.**

(a) How many of them do not repeat a digit?

We apply the multiplication principle

the process is to choose the digits in order

from left to right. We get

$\rightarrow 9 \cdot 9 \cdot 8 \cdot 7 \cdot 6$

the first place cannot be 0

(b) How many of them do not contain a 5 or a 6?

Same process as last time. We get

$7 \cdot 8 \cdot 8 \cdot 8 \cdot 8$

Again the first position is not zero.

(c) How many of them contain exactly one 2?

Two cases: 1 The 2 is first entry

2 The 2 is in the ~~third~~ second, third, fourth or fifth position.

Case 1 has

$9 \cdot 9 \cdot 9 \cdot 9$

possible numbers.

Case 2 has

$4 \cdot 8 \cdot 9 \cdot 9 \cdot 9$ possible numbers

# of positions for 2

# of ways to choose the first spot.

# of ways to choose the rest

Total is $9 \cdot 9 \cdot 9 \cdot 9 + 4 \cdot 8 \cdot 9 \cdot 9 \cdot 9$ by the addition principle