Version 1

MATH 61: INTRODUCTION TO DISCRETE STRUCTURES
MIDTERM #1

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Section 1F
Good Luck! Be sure to justify your answers!
No calculators, books or notes are allowed.

Problem	Points	Score
1	15	10
2	15	13
3	15	15
4	15	5
5	15	10
Total	75	53

Be careful, there are problems on both sides of the paper!

Date: April 18, 2016.

(1) (15 points) Prove by induction that for all $n \geq 4$, $2^n \geq n^2$.

base case

$n = 4$

$2^4 \geq 4^2$

$16 \geq 16$ ✓

inductive case

$2^{n+1} = 2^n \cdot 2$

$2^n \cdot 2 \geq n^2 \cdot 2$

we know that for the base case ?

$2^n \geq n^2$

if the left side is $\geq$ the right side — ?
adding an additional "scaling factor" of
two will maintain that property —

thus $2^n \geq n^2$ for all $n \geq 4$

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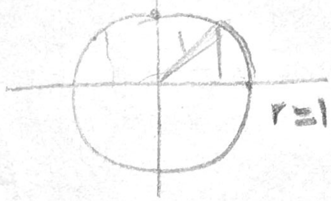
(2) (15 points) Consider the set A of points on the circle of radius 1 centered at the origin. Define a relation R on A by $(p, q) \in R$ exactly when q can be obtained by moving p some multiple of 45 degrees around the circle *in either direction!*. So for example $(1, 0)$ is related to $(-\sqrt{2}, \sqrt{2})$ since we can move from the first point to the second by moving counterclockwise by 45 degrees *three times*. However, $(0, 1)$ is not related to $(.5, \sqrt{1 - .5^2})$, because no amount of rotating by 45 degrees will move us between the points.

(a) Show that R is an equivalence relation.

$\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}$



$r=1$

1) reflexive

for any coordinate

1/3 $(x,y)\ R\ (x',y')$?

$x = x$ and $y = y$ so the relation is reflexive → it's the same point on the circle

2) symmetric

$(x,y)\ R\ (y,x)$

3/3 if (x,y) leads to a rotation that's a multiple of 45 degrees, then (y,x) will lead to that same rotation in the opposite direction, still divisible by 45, thus R is symmetric

3) transitive $(x,y)\ R\ (y,z)$ $(y,z)\ R\ (u,v)$

3/3 If the first point (x,y) is a multiple of 45 away from the second point (y,z) and the second point (y,z) is a multiple of 45 away from the third point, adding these two angles together will result in a sum that is still divisible by 45 - thus R is transitive

(b) List the elements of $[(1, 0)]$.

6/6 $\{(1,0), (-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$ $(0,1)$ $(-1,0)$, $(0,-1)$, $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}), (-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}), (\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})\}$

13

(3) (15 points; each of (a), (b) and (c) are weighted equally) Answer the following questions.

(a) Let $A = \{a, b, c\}$, $B = \{1, 2, 3\}$ and $C = \{b, c, 2, 3\}$. Determine each of the following. If it is a set, then write the set by listing the elements. If it is a number, then write which one. If it is a statement, then write true or false.

(i) $|\overline{B \cup C}|$

assuming the universal set is $A \cup B \cup C$

$\{a\}$

(ii) $A \times B$

$\{(a,1),(a,2),(a,3),(b,1),(b,2),(b,3),(c,1),(c,2),(c,3)\}$

(iii) $(A \cup B) - C$

$\{a, 1\}$

(iv) $C \cap (B \times B)$

$\emptyset$

(v) $(A \cap B) \subseteq C$

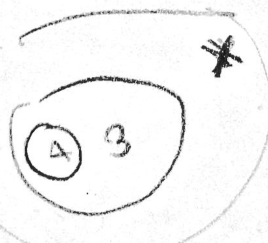
true

(b) Let A_1 and A_2 be sets. Find *disjoint* sets B_1 and B_2 such that $A_1 \cup A_2 = B_1 \cup B_2$.

$B_1 = A_1$

$B_2 = A_2 - A_1$

(c) Prove or disprove the following statement. For any sets A, B and X, if $A \subseteq B \subseteq X$, then $X - A \subseteq X - B$.



let's say A is $\{1\}$

B is $\{1, 2\}$

X is $\{1, 2, 3\}$

$X - A = \{2, 3\}$

$X - B = \{3\}$

X-A contains elements that are not in X-B, thus X-A is not a subset of X-B

(4) (15 points) Define the following sets:

(a) Let X be the set of strings with alphabet $\{0, 1\}$.

(b) Let Y be the set of strings with alphabet $\{a, b\}$.

(c) Let Z be the set of strings with alphabet $\{0, 1, a, b\}$.

Define a function $F : X \times Y \to Z$ by $F(\alpha, \beta) = \alpha \frown \beta$. Recall that $\alpha \frown \beta$ is the concatenation of α and β.

(a) Is F injective? Justify your answer. one to one

$\cancel{X \times Y = \{(0,a),(0,b),(1,a),(1,b)\}}$

~~Yes, for each string in the codomain that the function outputs there is only one entry in the domain that corresponds to it~~ Bad explanation.

1

(b) Is F surjective? Justify your answer. onto

No, it is not surjective. One counter example is the string 0ab10. The strings of the function are limited to a length of two. No!

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(c) Recall that the range of F is the set $\{z \in Z \mid \text{there is } (x, y) \in X \times Y \text{ such that } F(x, y) = z\}$. Show that $X \cup Y$ is a subset of the range of F.

$X \cup Y = \cancel{\{0,1,a,b\}}$

~~the function $F(x,y)$~~ results in a codomain of $\{0,1,a,b\}$. This is a subset of the range of F as we can take the empty set of X and concatenate it with Y - or vice versa leading to a single length string that is shown in $X \cup Y$?

1

(5) (15 points) Consider the numbers between 10000 and 99999 inclusive. **To receive full credit you must explain which counting principles you use and how you are using them.**

(a) How many of them do not repeat a digit?

5 digit places - multiplication principle

9 · 9 · 8 · 7 · 6

- first digit is limited to 9 digits rather than 10, after that its

(b) How many of them do not contain a 5 or a 6?

Total possibilities = 9 · 10 · 10 · 10 · 10 ← multiplication principle

Cases when a digits place will be 5

1 ·

Cases when a digits place will be 6

(c) How many of them contain exactly one 2?

Case when first digits place is 2

1 · 9 · 9 · 9 · 9

" second digits place is 2

8 · 1 · 9 · 9 · 9

third digits place is 2

8 · 9 · 1 · 9 · 9

fourth digits place is 2

8 · 9 · 9 · 1 · 9

fifth digits place is 2

8 · 9 · 9 · 9 · 1

multiplication principle when one place has to be 2

So using addition principle

$9^4 + 4(8 \cdot 9^3)$