1 Version 1

MATH 61: INTRODUCTION TO DISCRETE STRUCTURES
MIDTERM #1

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Section 1B

Good Luck! Be sure to justify your answers!
No calculators, books or notes are allowed.

Problem	Points	Score
1	15	8
2	15	15
3	15	15
4	15	7
5	15	3
Total	75	48

Be careful, there are problems on both sides of the paper!

Date: April 18, 2016.

(1) (15 points) Prove by induction that for all $n \geq 4$, $2^n \geq n^2$.

$P(n)$: $2^n \geq n^2$ for all $n \geq 4$

<u>Base case</u>: $n = 4$

$2^4 = 2 \cdot 2 \cdot 2 \cdot 2 = 16 \geq 16 = 4^2$. ✓

$\therefore P(4)$ is true.

<u>Inductive step</u>: Take an arbitrary but fixed n.

Assume $2^n \geq n^2$ is true.

Show for $n+1$, $2^{n+1} \geq (n+1)^2$ is also true.

$2^n \leq 2^{n+1} = 2^n \cdot 2 \geq \cancel{n^2 2}$ $\geq n^2 + 2n + 1 = (n+1)^2 \geq n^2$

$2(2^{n-1} + n) + 1$

$n^2 \geq 2^{n-1} + n + 1/2$

8/15

$n^2 \geq \frac{n^2}{2} + n + \frac{1}{2}$

$2^n \cdot 2 \geq n^2 + 2n + 1$

$2^n \cdot 2 - 2n - 1 \geq n^2$

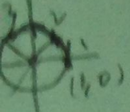
$2(2^n - n) - 1 \geq n^2$

?

$2(2^n - n) \geq n^2 + 1$

$2^n \geq \frac{n^2}{2} + n + 1/2 \leq n^2$

(2) (15 points) Consider the set A of points on the circle of radius 1 centered at the origin. Define a relation R on A by $(p,q) \in R$ exactly when q can be obtained by moving p some multiple of 45 degrees around the circle *in either direction!*. So for example $(1,0)$ is related to $(-\sqrt{2}, \sqrt{2})$ since we can move from the first point to the second by moving counterclockwise by 45 degrees *three times*. However, $(0,1)$ is not related to $(.5, \sqrt{1-.5^2})$, because no amount of rotating by 45 degrees will move us between the points.



$(p,q) \in R$ when q can be gotten by moving $p \pm k \cdot 45°$

(a) Show that R is an equivalence relation.

Is R reflexive? For any $p \in A$, is $(p,p) \in R$?
Yes because we can get to p by moving $0 \cdot 45°$! ✓

Is R symmetric? For any $(p,q) \in R$, is $(q,p) \in R$?
Yes because if one can get from p to q in $k \cdot 45°$, where $k \in \mathbb{Z}$, then one can get from q to p in $-k \cdot 45°$.
$\therefore (q,p) \in R$. ✓

Is R transitive? For any $(p,q) \in R$ and $(q,r) \in R$, is $(p,r) \in R$?
Yes, if we can get from p to q in $k \cdot 45°$ $(k \in \mathbb{Z})$ and from q to r in $l \cdot 45°$ $(l \in \mathbb{Z})$, then we can get from p to r in $(k+l) \cdot 45°$, where $k+l$ is still an integer because $k, l \in \mathbb{Z}$.
$\therefore (p,r) \in R$ and R is transitive. ✓

$\therefore R$ is an equivalence relation. ✓

(b) List the elements of $[(1,0)]$.

$[(1,0)] = \{y \mid ((1,0),y) \in R\}$
$= \{(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}), (-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}), (0,1), (-1,0), (-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}), (0,-1), (\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}), (1,0)\}$ ✓

15

(3) (15 points; each of (a), (b) and (c) are weighted equally) Answer the following questions.

(a) Let $A = \{a, b, c\}$, $B = \{1, 2, 3\}$ and $C = \{b, c, 2, 3\}$. Determine each of the following. If it is a set, then write the set by listing the elements. If it is a number, ~~then write~~ which one. If it is a statement, then write true or false.

(i) $|B \cup C| = 5$ (cardinality)

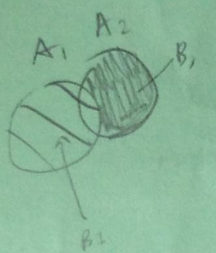
(ii) $A \times B = \{(a,1),(a,2),(a,3),(b,1),(b,2),(b,3),(c,1),(c,2),(c,3)\}$

(iii) $(A \cup B) - C = \{a, 1\}$

(iv) $C \cap (B \times B) = \emptyset$ because $B \times B = \{(1,1),(1,2),(1,3),\ldots\}$ is all ordered pairs, & there are no ordered pairs in C.

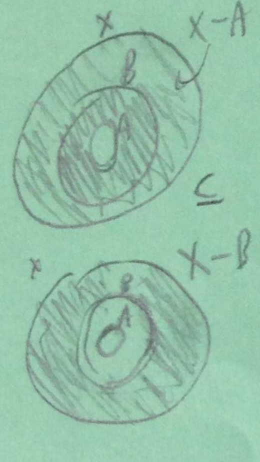
(v) $(A \cap B) \subseteq C$ True, empty set is a subset of any set.
$\emptyset \subseteq C$

(b) Let A_1 and A_2 be sets. Find *disjoint* sets B_1 and B_2 such that $A_1 \cup A_2 = B_1 \cup B_2$.



$B_1 = A_2 - A_1$
$B_2 = A_1$

(c) Prove or disprove the following statement. For any sets A, B and X, if $A \subseteq B \subseteq X$, then $X - A \subseteq X - B$.



This statement is $\boxed{\text{false}}$.
Suppose we have $X = \{1,2,3,4\}$, $B = \{1,2,3\}$, and $A = \{1\}$.
Then $X - A = \{2,3,4\}$
and $X - B = \{4\}$.
Since 2 and 3 $\in X - A$ but $2, 3 \notin X - B$,
$\therefore X - A \not\subseteq X - B$.

(4) (15 points) Define the following sets:

(a) Let X be the set of strings with alphabet $\{0, 1\}$. α

(b) Let Y be the set of strings with alphabet $\{a, b\}$. β

(c) Let Z be the set of strings with alphabet $\{0, 1, a, b\}$. F

Define a function $F : X \times Y \to Z$ by $F(\alpha, \beta) = \alpha \frown \beta$. Recall that $\alpha \frown \beta$ is the concatenation of α and β.

(a) Is F injective? Justify your answer.

1 to 1

Yes, F is injective. If we have $x, z \in X$ and $y, w \in Y$ and $F(x,y) = F(z,w)$, then $x = z$ and $y = w$. This is because for the concatenation of 2 strings to be equal at every position, the input strings x, y, z, w must be equal at every position. If either input string is different even in 1 position, that will lead to a different output concatenation from F. ok

✓

(b) Is F surjective? Justify your answer.

onto

0

F is ~~surjective~~ because it can result in strings that start with either 0, 1, a, and b. $F(\alpha, \beta)$ can start with a or b if α is an empty string. Similarly, $F(\alpha, \beta)$ can end with 0, 1, if β is passed as an empty string. Since $F(\alpha, \beta)$ concatenates variables instead of a fixed constant, it is surjective.

(c) Recall that the range of F is the set $\{z \in Z \mid \text{there is } (x, y) \in X \times Y \text{ such that } F(x, y) = z\}$. Show that $X \cup Y$ is a subset of the range of F.

$X \cup Y \subseteq$ range of F

~~We know $X \cup Y = \{0, 1, a, b\}$.~~ We must show every element is in the range of F.

2

Show $0 \in$ range of F.

If we let $x = 0$ and $y =$ '' (empty string) then $F(0, '') = 0 \frown '' = 0$. $\therefore \exists\ (x, y) = (0, '') \in X \times Y$ s.t. $F(0, '') = 0$. ✓

Show $1 \in$ range of F.

Similarly, if we let $x = 1$ and $y =$ '', then $F(1, '') = 1$. $\therefore \exists\ (x, y) = (1, '')$ s.t. $F(1, '') = 1$. ✓

$\therefore$ $a, b \in$ range of F

Let $x =$ '' and $y = a$ or b, then $F('', a) = a$ and $F('', b) = b$. $\exists\ (x, y) = ('', a)$ or $(x, y) = ('', b)$ s.t. $F('', a) = a$ and $F('', b) = b$. ✓

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(5) (15 points) Consider the numbers between 10000 and 99999 inclusive. **To receive full credit you must explain which counting principles you use and how you are using them.**

(a) How many of them do not repeat a digit?

digits 0 to 9
10 choices

1 2 3 4 5

$\boxed{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6} = 90 \cdot 56 \cdot 6 = 540 \cdot 56 = 3024$

I used the multiplication principle. Since there can be no repeated digits and there are 10 digs. total, each subsequent step's # choices decreases by one.

+2

(b) How many of them do not contain a 5 or a 6?

Only 8 choices

$8 \cdot 8 \cdot 8 \cdot 8 \cdot 8$

8^5

multiplication principle. each step only has 8 choices since 5 and 6 are excluded.

+1

(c) How many of them contain exactly one 2?

1 9 9 9 9

9^4

multiplication principle. One digit has one choice – 2. each other has 9 choices because 2 must be excluded since it occurs only once.