MATH 61: INTRODUCTION TO DISCRETE STRUCTURES
FINAL EXAM

INSTRUCTOR: SPENCER UNGER

Name: Solutions
ID # ____
Section ____
Good Luck! Be sure to justify your answers!
No calculators, books or notes are allowed.

Problem	Points	Score
1	15	
2	15	
3	15	
4	15	
5	15	
6	15	
7	15	
8	15	
9	15	
10	15	
Total	150	

Date: June 7th, 2016.

(1) (15 points total) Let S be a binary relation on $\mathbb{R} - \{0\}$ defined by $(x, y) \in S$ exactly when $\frac{x}{y} \in \mathbb{Q}$.

(a) (10 points) Show that S is an equivalence relation.

For any $x \neq 0$ $\quad \frac{x}{x} = 1 \in \mathbb{Q}$ So $(x,x) \in S$

For any $x, y \neq 0$ if $\frac{x}{y} \in \mathbb{Q}$ then $\frac{y}{x} = \frac{1}{\left(\frac{x}{y}\right)} \in \mathbb{Q}$

So $(x,y) \in S \to (y,x) \in S$.

For any $x, y, z \neq 0$, if $(x,y) \in S$ and $(y,z) \in S$

then $\frac{x}{y}, \frac{y}{z} \in \mathbb{Q}$. So $\frac{x}{z} = \frac{x}{y} \cdot \frac{y}{z} \in \mathbb{Q}$

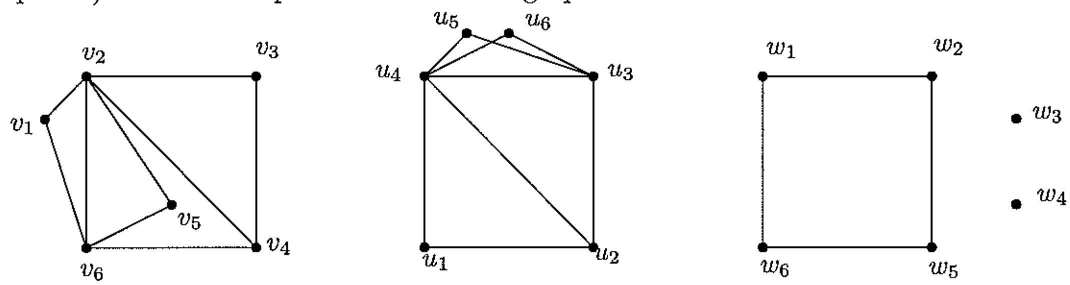
$\longrightarrow (x,z) \in S$.

(b) (5 points) Prove or disprove that for any $x \in \mathbb{R} - \{0\}$, $[x] = [2x]$.

To show $[x] = [2x]$ it is enough to prove

$(x, 2x) \in S$. but $\frac{x}{2x} = \frac{1}{2} \in \mathbb{Q}$ so it's true.

(2) (15 points) Answer the questions about the graphs below.



H_1 H_2 H_3

(a) Is H_1 isomorphic to H_2? If yes, exhibit an isomorphism. If no, justify your answer.

$V_4 \rightarrow U_2$ $V_6 \rightarrow U_3$
$V_3 \rightarrow U_1$ $V_5 \rightarrow U_5$
$V_2 \rightarrow U_4$ $V_1 \rightarrow U_6$

(b) Is H_3 isomorphic to H_2? If yes, exhibit an isomorphism. If no, justify your answer.

No they have a different number of edges.

(c) Is H_3 isomorphic to H_1? If yes, exhibit an isomorphism. If no, justify your answer.

No Same as before.

(d) Consider the following definition. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be graphs. We say that G_1 *embeds* in to G_2 if there is an injective function $f : V_1 \to V_2$ such that if $\{x, y\} \in E_1$, then $\{f(x), f(y)\} \in E_2$. Does H_1 embed in to H_2? Justify your answer by producing a function or explaining why one does not exist.

~~No H_3 has fewer edges.~~

Yes an isomorphism works for this.

(e) Using the definition from the previous part, does H_3 embed in to H_2? Justify your answer.

$W_1 \rightarrow U_4$ $W_3 \rightarrow U_5$
$W_2 \rightarrow U_3$ $W_4 \rightarrow U_6$
$W_5 \rightarrow U_2$
$W_6 \rightarrow U_1$

(3) (15 points) Let W_1 be the set of strings with alphabet $\{0, 1, 2\}$ and let W_2 be the set of strings with alphabet $\{0, 2\}$. Define a function $g : W_1 \to W_2$ by $g(\alpha)$ is the unique string obtained by removing all instances of 1 from α.

(a) Prove or disprove that g is one-to-one.

No $g(01) = g(0) = 0$, but $01 \neq 0$.

(b) Prove or disprove that g is onto.

Yes for any $\alpha \in W_2$, $\alpha \in W_1$ and

$g(\alpha) = \alpha$.

(c) Prove or disprove that $g \circ g = g$.

True. Suppose $\alpha \in W_1$.

$g(\alpha)$ has no 1's in it so $g(g(\alpha)) = g(\alpha)$

because there are no 1's to remove.

(4) (15 points total) Define a function $F : \mathbb{Z} \to \mathbb{Z}$ by $F(x) = x + 5$. Define a simple graph G with vertex set $\mathbb{Z}$ by $\{x, y\} \in E$ exactly when $F(x) = y$ or $F(y) = x$.

(a) For a given $x \in \mathbb{Z}$, what is the degree of x in G? No justification is required.

2

(b) Give an example of a path of length 3 in G. No justification required.

0, 5, 15, 20

(c) Determine the set of x which are connected to 16 in G. Write your answer in proper set notation without referencing anything from graph theory. No justification required.

$\{x \mid x = 1 + 5k \text{ for } k \in \mathbb{Z}\}$

(d) How many connected components does the graph G from the previous part have? No justification required.

5

(e) Define a function $F' : \mathbb{Z} \to \mathbb{Z}$ by $F'(x) = x + 10$ and define a graph G' using F' as we did for G from F. Is G' a subgraph of G? Justify your answer.

No $\{0, 10\}$ is an edge in G' but not in G.

(5) (15 points total) Circle T or F to indicate whether each statement is true or false. You do not need to justify your answers.

(a) **T** or **F.** $\begin{pmatrix} 0 & 2 & 1 \\ 2 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ is the adjacency matrix of a simple graph.

(b) **T** or **F.** In a weighted graph there is a unique shortest path between any two vertices.

(c) **T** or **F.** Let G be a graph with vertex set $\{1, 2, \ldots n\}$ and let A be the matrix whose i, j entry is the number of edges from vertex i to vertex j. For every $n \geq 0$, the i, j entry of A^n is the number of paths of length n from vertex i to vertex j.

(d) **T** or **F.** The shortest path algorithm works in any weighted graph where some of the weights are negative.

(e) **T** or **F.** If S is a set of cardinality n, then the cardinality of the powerset of S is $n!$.

(f) **T** or **F.** Prim's algorithm can be used to find a minimal spanning tree in a weighted graph where each weight is non-negative.

(g) **T** or **F.** The inclusion-exclusion principle for two sets A and B states that $|A \cup B| = |A| + |B| - |A \cap B|$.

(6) (15 points) Prove that if G is a connected and acyclic graph on n vertices, then G has $n-1$ edges.

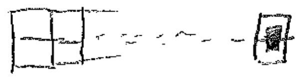
See class notes or the book.

(7) (15 points) Recall from class that a *proper tiling* of an $m \times n$ grid is a way of placing tiles so that (1) every square is covered and (2) no two tiles overlap. Let T_n be the number of ways to properly tile a $2 \times n$ grid with 2×2 tiles and 2×1 tiles. Recall that 2×1 tiles can be turned 90 degrees and used as 1×2 tiles.

Derive a recurrence relation for T_n. Don't forget the initial condition(s).

$T_1 = 1$, $T_2 = 3$

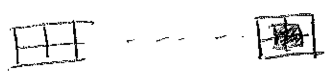
For T_n there are 3 cases

(1) The tiling ends with 

There are T_{n-1} tilings in this case

(2) The tiling ends with 

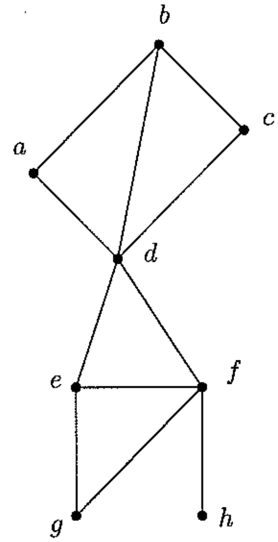
There are T_{n-2} tilings in this case

(3) The tiling ends with 

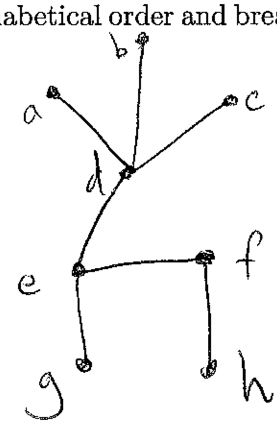
There are T_{n-2} tilings in this case.

So $T_n = T_{n-1} + 2T_{n-2}$.

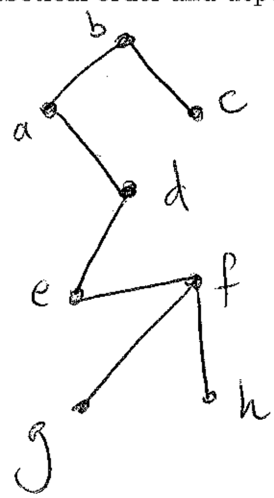
(8) (15 points) Answer the questions using the graph below.



(a) (7 points) Use alphabetical order and breadth first search starting at e to find a spanning tree.



(b) (8 points) Use alphabetical order and depth first search starting at d to find a spanning tree.



(9) (15 points) Recall that in a standard deck of cards there are 52 cards each of which has a suit and a face value. Possible face values are $A, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K$ and possible suits are $\diamondsuit, \heartsuit, \clubsuit, \spadesuit$. Any combination of suit and face value makes a card.

(a) How many ways are there to distribute the cards to 4 players, so that each player has 13 cards?

$$\binom{52}{13} \cdot \binom{39}{13} \cdot \binom{26}{13}$$

(b) A *run of 6 cards* is a set of 6 cards with the same suit and consecutive face values. For example the cards with suit $\heartsuit$ and face values $2, 3, 4, 5, 6, 7$ form a run of 6. How many ways are there to choose two runs of 6 cards from the full deck?

Two cases runs of the same suit or not.

Same suit: $\underbrace{4}_{\text{choose suit.}} \cdot 3$ ← ways to pick two runs of 6 from the same suit.

Different suits: $\underbrace{\binom{4}{2}}_{\text{choose 2 suits}} \cdot 8 \cdot 8$ ← number of ways to choose a run of 6 in one suit.

$\boxed{\text{total } 4 \cdot 3 + \binom{4}{2} \cdot 8 \cdot 8}$

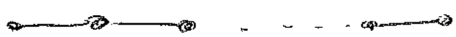
(c) Prove that if you distribute the cards as in part (a), then one of the players has at least 4 cards with a $\heartsuit$.

If all 4 players got at most 3 ♡'s then there would be 12 total, but 13 must be given out.

(10) (15 points) A *Hamilton path* is a path which touches every vertex. Which trees have Hamilton paths? Notice that there are two things to prove. First, the trees that you describe have Hamilton paths. Second, the trees that you describe are *the only* ones that have Hamilton paths.

They are exactly the simple paths of length n where $n \geq 0$.

(1) If we have a simple path of length n



then the whole graph is a Hamilton path.

(2) Suppose we have a tree with a Hamilton path. Draw a picture of the path

All vtxs are present so there can't be any other edges. More edges would create a cycle which ~~[illegible]~~ can't happen.