(1) (15 points) Prove by induction that for all $n \geq 4$, $2^n \geq n^2$.

Base case: For $n = 4$, $2^4 = 16$, $4^2 = 16$,

so $2^4 \geq 4^2$ and $2^n \geq n^2$ for $n = 4$.

Inductive step: Assume that for $n \geq 4$, $2^n \geq n^2$.

Then, for $n+1$, $2^{n+1} \geq (n+1)^2$,

as shown below.

$2^{n+1} \geq n^2(2)$, $(n+1)^2 = n^2 + 2n + 1$

We observe that $2n^2 \geq n^2 + 2n + 1$,

since $n^2 \geq 2n+1$ when $n \geq 4$.

Therefore, $2^{n+1} \geq 2n^2 \geq (n+1)^2$,

and so $2^{n+1} \geq (n+1)^2$

We conclude that for all $n \geq 4$, $2^n \geq n^2$.

(2) (15 points) Consider the set A of points on the circle of radius 1 centered at the origin. Define a relation R on A by $(p, q) \in R$ exactly when q can be obtained by moving p some multiple of 45 degrees around the circle *in either direction!*. So for example $(1, 0)$ is related to $(-\sqrt{2}, \sqrt{2})$ since we can move from the first point to the second by moving counterclockwise by 45 degrees *three times*. However, $(0, 1)$ is not related to $(.5, \sqrt{1 - .5^2})$, because no amount of rotating by 45 degrees will move us between the points.

(a) Show that R is an equivalence relation.

reflexive: for all $(x,y) \in R$, $(x,y)\,R\,(x,y)$

since R is defined by a rotation of a multiple of 45°. This means that a rotation of 0° is valid, so $(x,y)R(x,y)$ can be achieved with that transformation. ✓

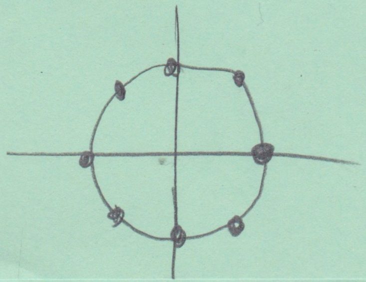
symmetric: if $(x,y)\,R(x',y')$, then $(x',y')\,R(x,y)$. This is because we know that (x',y') can be generated by rotating (x,y) a certain multiple of 45°, and so (x,y) can be generated by a rotation of the same # of degrees from (x',y'), but in the opposite direction. ✓

transitive: If $(x,y)\,R(x',y')$ and $(x',y')R(x'',y'')$, then $(x,y)\,R\,(x'',y'')$. This is because both of the initial relationships are related by a certain multiple of 45°, the operation performed on (x,y) to get to (x',y') and the operation performed on (x',y') to get to (x'',y'') can be performed consequently on (x,y) to get (x'',y'') and still remain a multiple of ✓ 45°.

(b) List the elements of $[(1, 0)]$.

[ ] ⇒ equivalence class

$[(1,0)] \to (1,0), \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right), (0,1), \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right),$
$(-1,0), \left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right), (0,-1), \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$ ✓



15

(3) (15 points; each of (a), (b) and (c) are weighted equally) Answer the following questions.

(a) Let $A = \{a, b, c\}$, $B = \{1, 2, 3\}$ and $C = \{b, c, 2, 3\}$. Determine each of the following. If it is a set, then write the set by listing the elements. If it is a number, then write which one. If it is a statement, then write true or false.

(i) $|B \cup C|$ $= 5$

1, 2, 3, b, c

(ii) $A \times B$ $\{(a,1), (a,2), (a,3), (b,1), (b,2), (b,3), (c,1), (c,2), (c,3)\}$

✓

$a \cup b = abc123$ (iii) $(A \cup B) - C$ $= \{a, 1\}$ 3/5

$-C = a\,1$

✗ (iv) $C \cap (B \times B)$ $\{b, 1, 2, 3, (1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3)\}$?

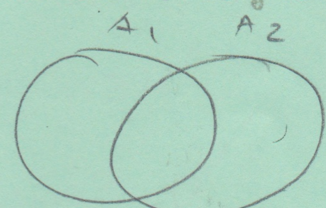
$A \cap B = \{a, b, c, 1, 2, 3\}$

✗ (v) $(A \cap B) \subseteq C$ False

(b) Let A_1 and A_2 be sets. Find *disjoint* sets B_1 and B_2 such that $A_1 \cup A_2 = B_1 \cup B_2$.

$B_1 = A_1$

$B_2 = A_2 - [A_1 \cap A_2]$



✓

(c) Prove or disprove the following statement. For any sets A, B and X, if $A \subseteq B \subseteq X$, then $X - A \subseteq X - B$.

False. By counter example $A = \{1, 2\}$

$B = \{1, 2, 3\}$

$X = \{1, 2, 3, 4\}$

✓

$X - A = \{3, 4\}$

$X - B = \{4\}$

$\rightarrow \{3, 4\} \not\subseteq \{4\}$

(4) (15 points) Define the following sets:

(a) Let X be the set of strings with alphabet $\{0, 1\}$.
(b) Let Y be the set of strings with alphabet $\{a, b\}$.
(c) Let Z be the set of strings with alphabet $\{0, 1, a, b\}$.

Define a function $F : X \times Y \to Z$ by $F(\alpha, \beta) = \alpha \frown \beta$. Recall that $\alpha \frown \beta$ is the concatenation of α and β.

(a) Is F injective? Justify your answer.

Yes. F is injective if $\alpha_1 \neq \alpha_2$ & $\beta_1 \neq \beta_2$, then $f(\alpha_1, \beta_1) \neq f(\alpha_2, \beta_2)$
If $\alpha_1 \neq \alpha_2$ & $\beta_1 \neq \beta_2$, it is impossible to have $f(\alpha_1, \beta_1) = f(\alpha_2, \beta_2)$ since α and β come from disjoint sets.

(b) Is F surjective? Justify your answer.

No. By counterexample, it is impossible for $f(\alpha, \beta)$ to generate the string abol, since β is concatenated to the end of α and β ~~can~~ only have the characters 'a' and 'b'.

(c) Recall that the range of F is the set $\{z \in Z \mid$ there is $(x, y) \in X \times Y$ such that $F(x, y) = z\}$. Show that $X \cup Y$ is a subset of the range of F.

Range of F in ~~set~~ of $\{0, 1, a, b\}$ contains strings with just $\{0, 1\}$ and $\{a, b\}$ and a combination of both.

range of F = $X \cup Y \cup$ (combination of strings from X and Y),

so $X \cup Y \subseteq$ range of F.

contains elements where $\alpha \in X$ and β = empty string and $\beta \in Y$ and α = empty string

this.

(5) (15 points) Consider the numbers between 10000 and 99999 inclusive. **To receive full credit you must explain which counting principles you use and how you are using them.**

(a) How many of them do not repeat a digit?

_ _ _ _ _

1-9   0-10

$9 \times 9 \times 8 \times 7 \times 6 = \boxed{27192}$

(by multiplication principle)

$81 \times 8 = 648$, $648 \times 7 = 4536$, $4536 \times 6 = 27192$

64×64: 256, 3840, 4096

(b) How many of them do not contain a 5 or a 6?

~~answer = (total #s) − (#'s containing 5 or 6) (by inclusion-exclusion)~~

~~total #s = 90000 #'s containing 5 or 6 =~~

$4096 \times 7 = 28602$

_ _ _ _ _

1, 2, 3, 4, 7, 8, 9   0, 1, 2, 3, 4, 7, 8, 9

$\Rightarrow 7 \times 8 \times 8 \times 8 \times 8$ ← multiplication principle

$= \boxed{28602}$

(c) How many of them contain exactly one 2?

_ _ _ _ _

multiplication principle

case 2 _ _ _ _ : $1 \times 9 \times 9 \times 9 \times 9 = 6561$

case _ 2 _ _ _ : $8 \times 1 \times 9 \times 9 \times 9 = 5832$

case _ _ 2 _ _ : $8 \times 9 \times 1 \times 9 \times 9 = 5832$

case _ _ _ 2 _ : $8 \times 9 \times 9 \times 1 \times 9 = 5832$

case _ _ _ _ 2 : $8 \times 9 \times 9 \times 9 \times 1 = 5832$

addition principle ← $6561 + 5832 + 5832 + 5832 + 5832 = \boxed{29889}$

81×81: 81, 648, 6561

$729 \times 8 = 5832$

$5832 \times 5 = 29160$, $+ 729 = 29889$