(1) (15 points) Consider the set $A$ of points on the circle of radius 1 centered at the origin. Define a relation $R$ on $A$ by $(p, q) \in R$ exactly when $q$ can be obtained by moving $p$ some multiple of 45 degrees around the circle *in either direction!*. So for example $(1, 0)$ is related to $(-\sqrt{2}, \sqrt{2})$ since we can move from the first point to the second by moving counterclockwise by 45 degrees *three times*. However, $(0, 1)$ is not related to $(.5, \sqrt{1 - .5^2})$, because no amount of rotating by 45 degrees will move us between the points.


(a) Show that $R$ is an equivalence relation.

Reflexive: $(p,q) \in R$ | p can be moved by some multiple of 45°.

$(p,p) \in R$ when p is moved 8 times w/ 45° intervals (total 360°).

Symmetric: if $(p,q) \in R$, then $(q,p) \in R$

This is true because if $p + 45° \cdot n = q$, then $q + (-1)(45° \cdot n) = p$, when n is the # of times you move.

Transitive: if $(p,x) \in R$ and $(x,q) \in R$, then $(p,q) \in R$.

Consider that ① $p + 45 \cdot n_1 = x$, where n is the # of moves, and that ② $x + 45 \cdot n_2 = q$. When substituting ① into ②, it becomes $p + 45 n_1 + 45 n_2 = q$. It is transitive because $(p,q) \in R$ is still distanced by a multiple of 45°.

(b) List the elements of $[(1, 0)]$.

$\{(1,0), (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}), (0,1), (-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}), (-1,0), (\frac{-\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}),$

$(0,-1), (\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})\}$



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(2) (15 points, each of (a), (b) and (c) are weighted equally) Answer the following questions.

(a) Let $A = \{a, b, c\}$, $B = \{1, 2, 3\}$ and $C = \{b, c, 2, 3\}$. Determine each of the following. If it is a set, then write the set by listing the elements. If it is a number, then write which one. If it is a statement, then write true or false.

(i) $|B \cup C|$ $\quad B \cup C = \{1, b, c, 2, 3\}$ $\quad |B \cup C| = \boxed{5}$

(ii) $A \times B$ $\quad \{(a,1), (a,2), (a,3), (b,1), (b,2), (b,3), (c,1), (c,2), (c,3)\}$

(iii) $(A \cup B) - C$ $\quad A \cup B = \{a, b, c, 1, 2, 3\}$

$\boxed{A \cup B - C = \{a, 1\}}$

(iv) $C \cap (B \times B)$

$\emptyset$

(v) $(A \cap B) \subseteq C$ $\quad (A \cap B) = \{\emptyset\}$

$\boxed{\text{True}}$

(b) Let $A_1$ and $A_2$ be sets. Find disjoint sets $B_1$ and $B_2$ such that $A_1 \cup A_2 = B_1 \cup B_2$.

$B_1 = A_1 - (A_1 \cap A_2)$

$B_2 = A_2 - (A_1 \cap A_2)$

what if $A_1 = A_2$

0/5

(c) Prove or disprove the following statement. For any sets $A, B$ and $X$, if $A \subseteq B \subseteq X$, then $X - A \subseteq X - B$.

if $A \subseteq B$, then $\{x \mid x \in A$ then $x \in B\}$

if $B \subseteq X$, then $\{x \mid x \in B$ then $x \in X\}$

Assume set $A = \{1, 2, 3\}$

set $B = \{1, 2, 3, 4\}$

set $X = \{1, 2, 3, 4, 5\}$

$X - A = \{4, 5\}$ $\quad$ In this example, there exists an element in

$X - B = \{5\}$ $\quad$ $X - A$ that is not in $X - B$, therefore,

$X - A$ is not a subset of $X - B$

(3) (15 points) Consider the numbers between 10000 and 99999 inclusive. To receive **full credit you must explain which counting principles you use and how you are using them.**

(a) How many of them do not repeat a digit?

5 digits  9 9 8 7 6

$\boxed{9 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \text{ numbers}}$

The multiplication princi is used to count which is not repeated

(b) How many of them do not contain a 5 or a 6?

7 8 8 8 8

$\boxed{7 \cdot 8 \cdot 8 \cdot 8 \cdot 8 \text{ numbers}}$

The multiplication princi is used to count the # that isn't either a 5 or a 6, which is 8 #.

(c) How many of them contain exactly one 2?

$1 \cdot 9 \cdot 9 \cdot 9 \cdot 9 + 8 \cdot 1 \cdot 9 \cdot 9 \cdot 9 + 8 \cdot 9 \cdot 1 \cdot 9 \cdot 9$ ...

$\boxed{9^4 + 4(8)(9)^3 \text{ \#s}}$

The multiplication princi is used to find how many # have exactly one 2 if 2 was in a set place. The Addition principle is later used to count if 2 exists in other places in the #

(4) (15 points) Define the following sets:

(a) Let $X$ be the set of strings with alphabet $\{0, 1\}$.

(b) Let $Y$ be the set of strings with alphabet $\{a, b\}$.

(c) Let $Z$ be the set of strings with alphabet $\{0, 1, a, b\}$.

Define a function $F : X \times Y \to Z$ by $F(\alpha, \beta) = \alpha \frown \beta$. Recall that $\alpha \frown \beta$ is the concatenation of $\alpha$ and $\beta$.

(a) Is $F$ injective? Justify your answer.

F = X×Y = {0a, 0b, 1a, 1b}

Yes, F is injective if F(α,β) = F(x,y), then αβ = xy.

(b) Is $F$ surjective? Justify your answer.

No, F is not surjective because of the existence of a0, a1, b0, etc. in Z.

Explain.

(c) Recall that the range of $F$ is the set $\{z \in Z \mid$ there is $(x, y) \in X \times Y$ such that $F(x, y) = z\}$. Show that $X \cup Y$ is a subset of the range of $F$.

X∪Y = {0, 1, a, b}

The range of F includes all possible set of strings of X×Y in Z. Because Z is defined as the set of strings with {0, 1, a, b}, X∪Y = Z. Therefore, there exist x ∈ X∪Y that x ∈ Z, so X∪Y is a subset of the range of F.

(5) (15 points) Prove by induction that for all $n \geq 4$, $2^n \geq n^2$.

Base Case: Show that $2^n \geq n^2$ when $n \geq 4$:

If $n = 4$: $2^4 \geq 4^2$

$16 \geq 16$ ✓

Base case is true.

Inductive step: Assume $n$ is a random but fixed # that is greater than or equal to 4

Assume $2^n \geq n^2$ for $n \geq 4$.

Show that $2^{n+1} \geq (n+1)^2$

$2^{n+1} = 2(2^n)$

$(n+1)^2 = (n+1)(n+1)$

$(2)(2^n) \geq (n+1)(n+1)$

Because it is established that $2^n \geq n^2$ for when $n \geq 4$, $(2)(2^n) \geq (n+1)(n+1)$ is true because the left side is inc by multiple of 2, while the right side is inc. by a multiple of $(n+1)$, when $n \geq 4$. $n+1$ is therefore always greater than 2.

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