Version 1

MATH 61: INTRODUCTION TO DISCRETE STRUCTURES
MIDTERM #1

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	Points	Score
	15	13
	15	14
3	15	15
4	15	12
5	15	10
Total	75	64

Be careful, there are problems on both sides of the paper!

Date: April 18, 2016.

(1) (15 points) Prove by induction that for all $n \geq 4$, $2^n \geq n^2$.

Base step: $n=4$

$(n+1)^2 = n^2+2n+1$

$2^4 = 16, \ 4^2=16$

$2^4 = 16 \geq 4^2 = 16$

Inductive step:

Assume that $2^n \geq n^2$. Show that $2^{n+1} \geq (n+1)^2$

$2^{n+1} = 2^n \cdot 2 \doteq 2(n^2) \geq ($

$2^n \geq n^2$

$2 \cdot 2^n \geq 2 \cdot n^2$

$2 \cdot 2^n + 4n + 2 \geq 2n^2 + 4n + 2$

$2 \cdot 2^n + 4n + 2 \geq 2(n+1)^2$

$2^{n+1} \geq 2(n+1)^2 - 4n + 2$

$2^{n+1} \geq 2((n+1)^2 - 2n + 1)$ close

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The answer above is wrong, but I wanted the following result,

show that $2^{n+1} \geq (n+1)(n+1)$

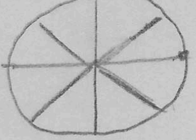
$2^{n+1} = 2 \cdot 2^n \geq 2 \cdot n^2 \quad \ldots \quad n^2 + 2n + 1 \geq (n+1)^2$

even closer.

So, by induction, $\forall n \geq 4$, and $n \in \mathbb{Z}$, $2^n \geq n^2$.

(2) (15 points) Consider the set A of points on the circle of radius 1 centered at the origin. Define a relation R on A by $(p, q) \in R$ exactly when q can be obtained by moving p some multiple of 45 degrees around the circle *in either direction!*. So for example $(1, 0)$ is related to $(-\sqrt{2}, \sqrt{2})$ since we can move from the first point to the second by moving counterclockwise by 45 degrees *three times*. However, $(0, 1)$ is not related to $(.5, \sqrt{1 - .5^2})$, because no amount of rotating by 45 degrees will move us between the points.

(a) Show that R is an equivalence relation.



Let $(x,y) \in A$. Then $((x,y),(x,y)) \in R$ because every point is a $0°$ rotation from itself. So, R is reflexive

Let $(x,y),(u,v) \in A$ such that $(x,y) R (u,v)$. Then, this means that (u,v) is a multiple of a $45°$ rotation from (x,y). So, order does not matter for this relation because we can get (u,v) by rotating (x,y) some multiple of $45°$.

$\Rightarrow ((u,v),(x,y)) \in R \Rightarrow R$ is symmetric.

Let $(x,y),(u,v),(z,w) \in A$ such that $(x,y) R (u,v)$ and $(u,v) R (z,w)$. Then (u,v) is a multiple of $45°$ rotation from (x,y), and (z,w) is a multiple of $45°$ rotation from (x,y). So, (z,w) must also be a multiple of $45°$ rotation from (x,y) because we can get to (u,v) from (x,y). Explain

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$\Rightarrow (x,y) R (z,w) \Rightarrow R$ is transitive.

So, R is an equivalence relation.

(b) List the elements of $[(1, 0)]$.

$[(1,0)] = \{(1,0), (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}), (0,1), (-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}), (-1,0), (-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}),$
$(0,-1), (\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\}$

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(3) (15 points; each of (a), (b) and (c) are weighted equally) Answer the following questions.

(a) Let $A = \{a, b, c\}$, $B = \{1, 2, 3\}$ and $C = \{b, c, 2, 3\}$. Determine each of the following. If it is a set, then write the set by listing the elements. If it is a number, then write which one. If it is a statement, then write true or false.

$B \cup C$
$= \{1, 2, 3, b, c\}$

(i) $|B \cup C|$ $\boxed{5}$

(ii) $A \times B$ $\{(a,1), (a,2), (a,3), (b,1), (b,2), (b,3), (c,1), (c,2), (c,3)\}$

(iii) $(A \cup B) - C$

$\{a, 1\}$

values in C and values in $B \times B$ $\longrightarrow$ (iv) $C \cap (B \times B)$

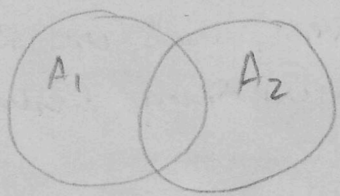
$\emptyset$

$A \cap B = \{\emptyset\}$

(v) $(A \cap B) \subseteq C$

True

(b) Let A_1 and A_2 be sets. Find *disjoint* sets B_1 and B_2 such that $A_1 \cup A_2 = B_1 \cup B_2$.



Let $B_1 = A_1$ and $B_2 = A_2 - A_1$.

Then $B_1 \cup B_2 = A_1 \cup A_2$

(c) Prove or disprove the following statement. For any sets A, B and X, if $A \subseteq B \subseteq X$, then $X - A \subseteq X - B$.

Let $A = \{1, 2, 3\}$, $B = \{1, 2, 3, 4\}$, $X = \{1, 2, 3, 4, 5\}$.

Then $A \subseteq B \subseteq X$, and

$X - A = \{4, 5\}$, $X - B = \{5\}$

But $X - A \not\subseteq X - B$

because $4 \in X - A$ but $4 \notin X - B$,

(4) (15 points) Define the following sets:

(a) Let X be the set of strings with alphabet $\{0, 1\}$.

(b) Let Y be the set of strings with alphabet $\{a, b\}$.

(c) Let Z be the set of strings with alphabet $\{0, 1, a, b\}$.

Define a function $F : X \times Y \to Z$ by $F(\alpha, \beta) = \alpha \frown \beta$. Recall that $\alpha \frown \beta$ is the concatenation of α and β.

(a) Is F injective? Justify your answer.

Yes. Take $\alpha_1, \alpha_2 \in X$, $\beta_1, \beta_2 \in Y$. If $(\alpha_1, \beta_1) \neq (\alpha_2, \beta_2)$, there must be a point of difference between α_1 and α_2 and/or β_1, β_2; either the lengths are different or the combination of the respective alphabet is different. So, concatenating α_1 with β_1 and α_2 with β_2 will preserve this dissimilarity. So if $(\alpha_1, \beta_1) \neq (\alpha_2, \beta_2)$ $F(\alpha_1, \beta_1) \neq F(\alpha_2, \beta_2) \Rightarrow F$ is injective.

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Not quite enough

(b) Is F surjective? Justify your answer.

No. Note that F is defined as $F(\alpha, \beta) = \alpha \frown \beta$, so all strings returned by this function must end with some combination of a's or b's. So, if we take $0101 \in Z$, there is no $\alpha \in X, \beta \in Y$ such that $F(\alpha, \beta) = 0101$.

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β = empty string

(c) Recall that the range of F is the set $\{z \in Z \mid \text{there is } (x, y) \in X \times Y \text{ such that } F(x, y) = z\}$. Show that $X \cup Y$ is a subset of the range of F.

If $X \cup Y \subseteq \text{range}(F)$, then $\forall \alpha \in X \cup Y$, $\alpha \in \text{range}(F)$

Take $\alpha \in X \cup Y$, so α is either a string made by the alphabet in X or by the alphabet in Y.

Case 1: $\alpha \in X$. Then take β = empty string so $F(\alpha, \text{empty string}) = \alpha$.

Case 2: $\alpha \notin X$. So then $\alpha \in Y$. Take β = empty string so $F(\text{empty string}, \beta) = \beta$.

Thus, for all elements in $X \cup Y$, they all exist in range(F) as well.

(5) (15 points) Consider the numbers between 10000 and 99999 inclusive. **To receive full credit you must explain which counting principles you use and how you are using them.**

(a) How many of them do not repeat a digit?

Using multiplication principle to find how many different possibilities,

$\underline{9}\ \underline{9}\ \underline{8}\ \underline{7}\ \underline{6} =$

9 because 0 cannot be first digit

$$\begin{array}{r} 81 \\ \cdot\ 56 \\ \hline 486 \\ 4050 \\ \hline 4536 \\ \cdot\ 6 \\ \hline 27216 \end{array}$$

27216 numbers

(b) How many of them do not contain a 5 or a 6?

Multiplication principle to restrict digit values from having a 5 or a 6

$\underline{7}\ \underline{8}\ \underline{8}\ \underline{8}\ \underline{8} = 7 \cdot 8^4$ possible numbers.

cannot contain a 0,5,6

(c) How many of them contain exactly one 2?

Use addition principle to count all 5 digit numbers with 2 in first digit, 2 in second digit... etc. Also use multiplication principle to find number of possibilities where the number 2 cannot repeat.