Math 61: Introduction to Discrete Structures
Midterm #1

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Problem	Points	Score
1	20	20
2	20	20
3	10	10
4	10	10
5	20	19
6	20	17
Total	100	96

1. (20 points) Show that for all $n \geq 1$, $8^n - 3^n$ is divisible by 5.

Base Case: $8^{(1)} - 3^{(1)} = 5$

Now, assume, $8^n - 3^n$ is %-able by 5

Show $8^{n+1} - 3^{n+1}$ also %-able

$(8^n - 3^n)8 \to$ mult by 8 divisible by 5

$8^{n+1} - 8 \times 3^n$

$= 8^{n+1} - 3^{n+1} - 5 \times 3^{n+1}$ still divisible by 5

if we add $(5 \times 3^{n+1})$ (%-able by 5)

the sum is still %-able by 5

$8^{n+1} - 3^{n+1} - 5 \times 3^{n+1} + 5 \times 3^{n+1}$ } all %-able by 5

$= 8^{n+1} - 3^{n+1}$ } %-able by 5

By law of induction,

for all $n \geq 1$, $8^n - 3^n$ is %-able by 5

20

2. (20 points)

(a) Throughout this problem use following sets:

$[-3, 18)$

$$\begin{aligned} A &= \{x \in \mathbb{R} \mid -3 \leq \lfloor x \rfloor \leq 17\} \\ B &= \{y \in \mathbb{Z} \mid -5 < y < 10\} \quad (-5, 10) \\ C &= \{z \in \mathbb{R} \mid z^2 \leq 100\} \quad [-10, 10] \\ D &= \{w \in \mathbb{R} \mid w < -9\} \quad (-\infty, -9) \\ E &= \{n \in \mathbb{N} \mid n^2 + 1 \text{ is even }\} \end{aligned}$$

For each of the following statements determine whether it is true or false. Just write T or F for each.

i. $A \subseteq B$

F (17.5)

ii. $C \cap D = \emptyset$

F $(-9.5)^2 \leq 100$

iii. $\{5\} \subseteq E \cap B$

T $5 \in B$ $5^2+1=26$

iv. $10 \in C - D$

T $10 \in C$ & $10 \notin D$ $10^2 = 100$ $10 > -9$

v. $17.5 \in A$

T

Parts (b) and (c) on the next page.

(b) Let $T = \{(m,n) \in \mathbb{N} \times \mathbb{N} \mid -10 \leq m \leq 10 \text{ and } -10 \leq n \leq 10\}$ and $S = \{(m,n) \in \mathbb{N} \times \mathbb{N} \mid m^2 + n^2 \leq 100\}$. Is it true that $S = T$? Justify your answer.

$(10, 10) \in T$

$10^2 + 10^2 = 200 > 100$

So $(10, 10) \notin S$

No, there's an element unique to T

So $T \neq S$

(c) Let X be a finite set. Give the definitions of both $\mathcal{P}(X)$ (the powerset of X) and $|X|$.

A is a set

$P(x) = \{A \mid A \subseteq X\}$ all subsets of X

$|X|$ = number of elements in X

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3. (10 points) Let $f : \mathbb{N} \times \mathbb{R} \to \mathbb{R}$ be given by $f(n, x) = nx$.

(a) Is f one-to-one? Justify your answer.

no

if $n=0$ & x can be any thing

$f(0, x) = 0 \cdot x = 0$

Ok.

(b) Is f onto? Justify your answer.

$\boxed{\text{yes}}$, ~~given any element $y \in \mathbb{R}$ and any fixed value $y \neq 0$ $v \in \mathbb{N}$, we can find an $x \in \mathbb{R}$ st. $vx = y$ by~~ $\cancel{x = \frac{y}{v} \in \mathbb{R}}$

~~if $y = 0$, we can use $x = 0$~~

Or take $n = 1 \in \mathbb{N}$

$1 \times$ any real # = itself $\in \mathbb{R}$

4. (10 points) Recall that for a real number x, $\lfloor x \rfloor$ is the greatest integer less than or equal to x. Define a function $g : \mathbb{R} \to \mathbb{R}$ by the formula $g(x) = \lfloor x + \frac{3}{2} \rfloor$. From a homework problem we know that the relation R on $\mathbb{R}$ given by $(x, y) \in R$ exactly when $g(x) = g(y)$ is an equivalence relation. (You don't need to show this.) Answer the following questions:

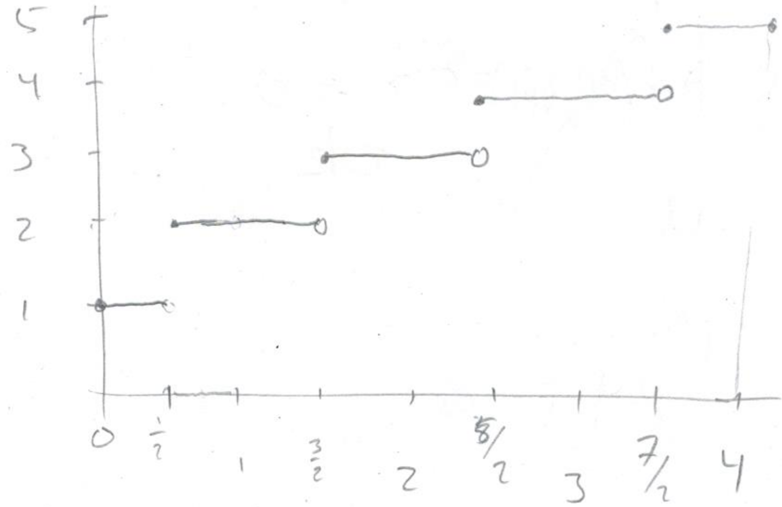
(a) Graph the function g on the interval $[0, 4]$.

$\frac{1}{2} + \frac{3}{2} = 2$

$1 + \frac{3}{2} = 2\frac{1}{2}$

$\frac{5}{2} = 2.5$

$\frac{5}{2} + \frac{3}{2} = 4$



(b) The equivalence class of 3, [3] is an interval on the real line. Which interval is it?

$[5/2, 7/2)$

5. (20 points) Given two strings α and β we say that α is an *initial segment* of β if there is a string γ such that $\alpha\gamma = \beta$. Recall that $\alpha\gamma$ is the concatenation of α and γ. For example 011 is an initial segment of 01101, but 10 is not an initial segment of 11010.

Let X be the set of binary strings of length at most 8. Define a relation R on X by (α, β) exactly when α is an initial segment of β. Answer the following questions about R. Be sure to justify your answers.

(a) Is R reflexive? (Hint: There is a string of length 0, an empty string.)

sure, for any string $\beta \in X$,

$\beta \lambda = \beta$

$\uparrow$ empty string

✓

(b) Is R symmetric?

✓

no

$\beta = 011$

$\alpha = 0$

$|\beta| > |\alpha|$, so β can't possibly be the initial segment

(c) Is R transitive?

is so if $(\alpha,\beta) \in R$ and $(\beta, \eta) \in R$ then $(\alpha, \eta) \in R$

✓ Yes

so if $(\alpha,\beta) \in R$, there's a γ st $\alpha\gamma = \beta$ and if $(\beta, \eta) \in R$, then there's a ε st $\beta\varepsilon = \eta$

So! $\Leftarrow$ $\alpha\gamma\varepsilon = \eta$ so $(\alpha, \eta) \in R$!

(d) Is R antisymmetric?

$\rightarrow$ if $(\alpha,\beta) \in R$ and $(\beta, \alpha) \in R$, then $\alpha = \beta$

yes, if $\alpha \neq \beta$, then $\alpha\gamma = \beta$, and $\gamma \neq \lambda$, therefore $|\beta| > |\alpha|$, (β is longer than α) so we can't possibly have β been the initial segment

why? There are two cases

0 —1

Contrapositive is the property of antisymmetric-ness

Proven if not $\alpha \neq \beta$ then not (α,β) and $(\beta,\alpha) \in R$

17

6. (20 points) Work with a standard deck of cards. Recall that there are 52 cards and each card has a suit ($\diamondsuit, \heartsuit, \clubsuit, \spadesuit$) and a face value ($1, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K$). All combinations of suits and face values are possible.

This problem will have you count the number of 5 card hands that have exactly 2 cards with face value 5 or exactly 3 cards with face value J. **For full credit please state any counting rules or principles that you use. You do not need to simplify your answers.**

face val 5 -> 4 card,

(a) Count the number of 5 card hands with exactly 2 cards of face value 5.

$$\binom{4}{2}\binom{48}{3}$$

multiplication principle

6

choose 2 5's

choose 3 not 5's

(b) Count the number of 5 card hands with exactly three cards with face value J.

$$\binom{4}{3}\binom{48}{2}$$

multiplication principle

6

(c) Using your previous answer count the number of 5 card hands with exactly 2 cards of face value 5 or exactly 3 cards which have face value J. (Hint 1: Your count should include the hands that have exactly 2 cards with face value 5 *and* exactly 3 cards with face value J. Hint 2: Be careful!)

5

$$\binom{4}{2}\binom{48}{3} + \binom{4}{3}\binom{48}{2} - \binom{4}{2}\binom{4}{3}$$

hands w/ both exactly 2 5's and 3 J's

inclusion-exclusion