# Math 61: Introduction to Discrete Structures
# Final Exam

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December 15th, 2014

Name: Solutions
ID # ____________
Section ________
Good Luck! Be sure to justify your answers!
No calculators, books or notes are allowed.

| Problem | Points | Score |
|---|---|---|
| 1 | 20 | |
| 2 | 20 | |
| 3 | 20 | |
| 4 | 20 | |
| 5 | 20 | |
| 6 | 20 | |
| 7 | 20 | |
| 8 | 20 | |
| 9 | 20 | |
| 10 | 20 | |
| Total | 200 | |

1. (20 points) Let $F : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ be given by $F(x, y) = (x + 1, y + 2)$. Answer the following questions about $F$. **Be sure to justify your answers!**

(a) Is $F$ onto?

Yes. Given $(x,y) \in \mathbb{Z} \times \mathbb{Z}$

$F(x-1, y-2) = (x,y)$

(b) Is $F$ one-to-one?

Yes. If $F(x,y) = F(w,z)$ then

$(x+1, y+2) = (w+1, z+2)$. So $x+1 = w+1$

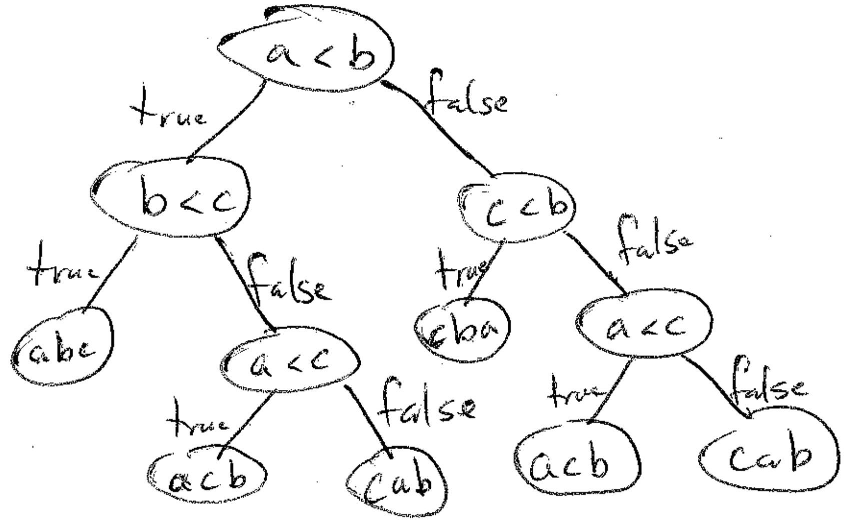
and $y+2 = z+2$. So $x = w$ and $y = z$.

(c) Define a simple graph $G$ with vertex set $\mathbb{Z} \times \mathbb{Z}$ such that $\{(x, y), (z, w)\} \in E$ if $F(x, y) = (z, w)$ or $F(z, w) = (x, y)$. Determine the set of $(x, y)$ which are connected to $(0, 0)$ in $G$. Write your answer in proper set notation. **The property you give that describes the set should not include anything from graph theory!**

$\{ (n, 2n) \mid n \in \mathbb{Z} \}$

2. (20 points) Answer the following questions:

(a) Write a binary decision tree to sort three numbers $a$, $b$ and $c$.




(b) What is the height of the tree that you drew?

3

(c) How many terminal nodes does your tree have?

$3! = 6$

(d) If $h$ is the height of your tree, then what is the maximum number of terminal nodes possible among all binary trees of height $h$?

$2^3$

3. (20 points) Define a relation $R$ on $\mathbb{R}$ by $(x, y) \in R$ if $x^2 + y^2 = 1$. Answer the following questions and be sure to justify your answers.

(a) Is $R$ reflexive?

No $(1,1) \notin R$ since $1^2+1^2=2 \neq 1$.

(b) List the elements of the set $\{x \mid (x, x) \in R\}$.

$(\sqrt{2}, \sqrt{2})$, $(-\sqrt{2}, -\sqrt{2})$ are in $R$

So $\sqrt{2}, -\sqrt{2}$ are in the specified set.

(c) Is $R$ symmetric?

Yes. if $(x,y) \in R$ then $x^2+y^2=1$

So $y^2+x^2=1$ which gives $(y,x) \in R$.

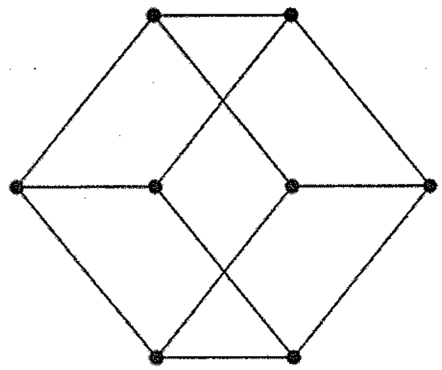
(d) Is $R$ transitive?

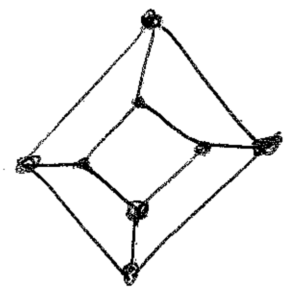
No $(1,0) \in R$ and $(0,1) \in R$

but $(1,1) \notin R$.

4. (20 points) Answer the following questions:

(a) Show that the graph below is planar.





(b) Verify Euler's formula for the graph above.

Formula is $v - e + f = 2$

$v = 8$

$e = 12$

$f = 6$

$8 - 12 + 6 = 2$

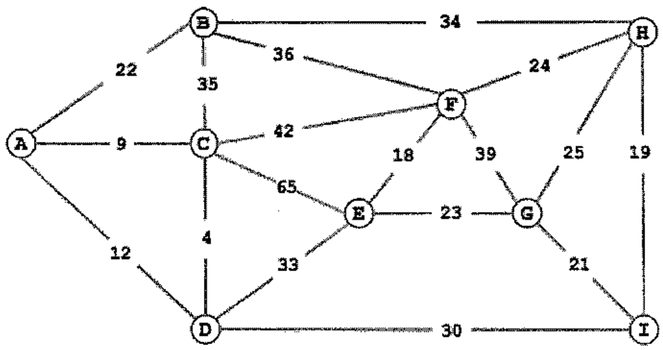
(c) Suppose that a graph $G$ is not planar. What does Kuratowski's theorem tell you about $G$?

It has a subgraph which is a subdivision of $K_{3,3}$ or $K_5$.

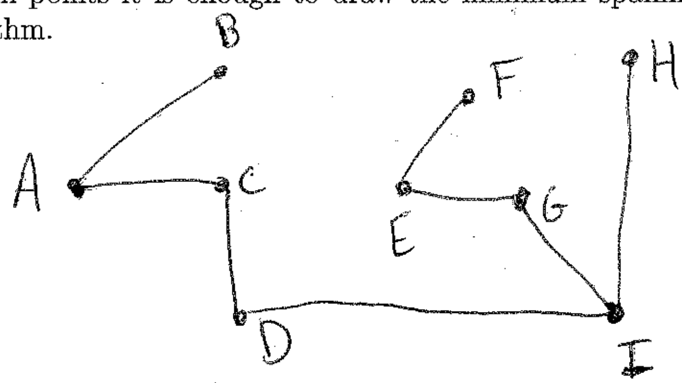
You could have written:

It has a subgraph homeomorphic to $K_{3,3}$ or $K_5$

5. (20 points) Answer the following questions about the weighted graph below:




(a) Use Prim's algorithm starting at the vertex $A$ to find a minimum spanning tree. For full points it is enough to draw the minimum spanning tree obtained by the algorithm.




(b) Use Djikstra's algorithm to find the shortest path from $D$ to $F$. Please write the order in which you visit the vertices and note that this is probably different from the shortest path!

D, C, A, I, E, B, F.

There is a mistake in this problem which we wrote on the board
It should be $V = \{0, 1, \cdots n-1\}$ .... for all $i < n-1$ ..... $\{0, n-1\}$
with the rest the same. Grading will be generous.

6. (20 points) Let $G$ be a graph that is a simple cycle of length $n$. Formally we set $V = \{1, 2, 3, \ldots n\}$ and for all $i < n$ we make $\{i, i+1\}$ an edge and also $\{0, n\}$ an edge. This problem will have you count the number of isomorphisms from $G$ to $G$.

   (a) If we are building an isomorphism $f$ up from nothing, then how many ways are there to choose a value for $f(0)$?

   There are $n$ possible values

   (b) Given that we have chosen a value for $f(0)$ how many ways are there to choose a value for $f(1)$?

   There are two possible values since $f(1)$ must be adjacent to $f(0)$ and every vertex has degree two.

   (c) Using the previous two parts (or otherwise), count the number of isomorphisms from $G$ to $G$. Be sure to explain any counting principles that you use.

   Choosing $f(0)$ and $f(1)$ completely determines the isomorphism. So by the multiplication principle there are $n \cdot 2$ isomorphisms from $G$ to $G$.

7. (20 points) Show that if $G$ is a connected, acyclic graph with $n$ vertices, then it has $n-1$ edges.

This was given in class, in the book and on the forum.

8. (20 points) Let $T_n$ be the sequence of numbers defined by the recurrence $T_n = T_{n-1} + 2n + 1$ with initial condition $T_0 = 0$. Obtain a guess for a closed formula for $T_n$ by repeatedly expanding the recurrence relation. Prove your guess by induction. (Hint: You may need the formula $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$.)

$$T_n = T_{n-1} + 2n+1$$

$$= T_{n-2} + 2(n-1)+1 + 2n+1$$

$$= T_{n-3} + 2(n-2) +1 + 2(n-1)+1 + 2n+1$$

$$\vdots$$

$$= T_0 + 2\left(\sum_{k=1}^{n} k\right) + n$$

$$= 0 + n(n+1) + n = n^2+2n$$

Proof by induction that $T_n = n^2+2n$ for all $n$.

Base Case: $n=0$ $\quad T_0 = 0^2+2\cdot 0 = 0$

Induction Step: Assume $T_n = n^2+2n$

$$T_{n+1} = T_n + 2(n+1)+1$$

$$= n^2+2n+2(n+1)+1 \quad \text{By the induction hypothesis.}$$

$$= n^2+2n+1 + 2(n+1)$$

$$= (n+1)^2+2(n+1) \quad \text{as required.}$$

9. (20 points) Let $S \subseteq \{n \mid 1 \leq n \leq 20\}$ be a fixed subset with $|S| = 7$. Answer the following questions:

(a) How many subsets of $S$ are there?

$2^7 = |\text{Powerset of } S|$

(b) Let $T \subseteq S$. What is the minimum possible value for the sum of all elements of $T$?

T could be $\emptyset$ so the sum of all elements of $\emptyset$ is 0.

(c) Let $T \subseteq S$. What is the maximum possible value for the sum of elements of $T$?

T could be equal to S when $S = \{20, 19, 18, 17, 16, 15, 14\}$

So in this case the sum of all elements of T is

$7 \cdot (20) - (1+2+3+4+5+6) = 140 - 21 = 119$.

(d) Prove that there are two different subsets $X$ and $Y$ of $S$ where the sum of the elements of $X$ is equal to the sum of the elements of $Y$. (Hint: Consider the function $H$ which takes a subset $T$ of $S$ and returns the sum of all elements of $T$.)

$H: \text{Powerset of } S \longrightarrow \{1, \dots 119\}$.

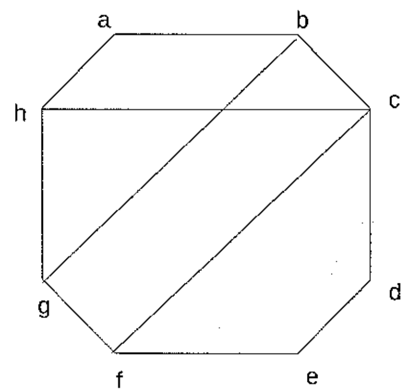
So since $2^7 = 128 > 119$

There are distinct X and Y such that $H(X) = H(Y)$

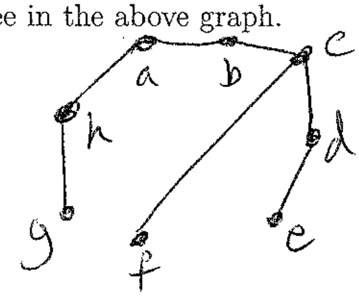


Which is what we wanted

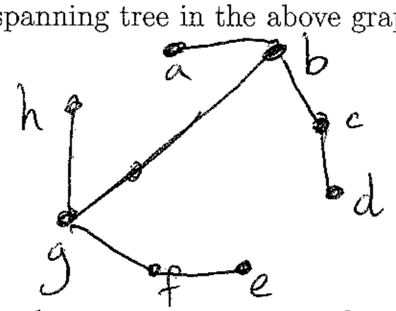
10. (20 points) Answer the questions about the graph below:




(a) Use depth first search starting at vertex $f$ and the alphabetical order on the vertices to find a spanning tree in the above graph.




(b) Use breadth first search starting at the vertex $g$ and the alphabetical order on the vertices to find a spanning tree in the above graph.




(c) Determine whether the two spanning trees that you found are isomorphic. If they are exhibit an isomorphism. If they are not, then explain why not.

The graph for (a) has exactly one vertex of degree 3

The graph for (b) has exactly two vertices of degree 3

So they are not isomorphic.