Discrete Structures
Math 61, Winter 2015 — Schaeffer
Midterm Exam 1

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In LARGE CAPITALS, the first 3 letters of your last/family name:

CHU

Circle your TA. If you do not know your TA's name, you must speak with Professor Schaeffer when you hand in your exam so he can look it up.

Zhu (A,B) Rosenbaum (C,D) Zhang (E,F)

Instructions: Complete all problems. Notes and electronics are not permitted. Good luck!

Problem	Notes	Grade
1	✓	7
2	✓	3
3	✓	4
4	✓	7
5	0	2
6	✓	4
7	✓	7
8	✓	4
Total		38

1. Consider the sets S, T, and U defined below:

$$S = \{ n \in \mathbb{Z} : \text{there is } m \in \mathbb{Z} \text{ such that } n = m^2 \}$$ $\{1, 4, 9, 16 \ldots\}$

$$T = \{ r \in \mathbb{R} : \text{there is } x \in \mathbb{R} \text{ such that } r = \sqrt{x} \}$$ $\{1, \sqrt{2}, \sqrt{3}, 2, \sqrt{5} \ldots\}$

$$U = \{ r \in \mathbb{R} : -2 < r < 2 \text{ and } r \neq 0 \}$$ $\{-1.9 \ldots, \cancel{0}, 1.\ldots\}$

Which of the following statements are true about these sets?

Circle all correct answers.

a. S and U are disjoint. False : share 1

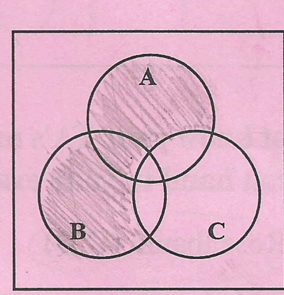
b. $S \subseteq T$

c. $S \cap T = S$

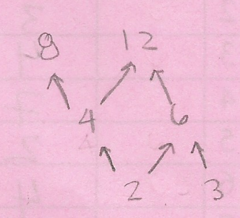
d. $T \cap U = U$ False b/c U can have neg # & T can't

e. $T \cup U = \{ r \in \mathbb{R} : r + 2 > 0 \}$

2. In the Venn diagram below, shade the region corresponding to $(A \cup B) \cap \overline{C}$.



3. Draw a Hasse diagram for the partial order of *divisibility* on the set $S = \{2, 3, 4, 6, 8, 12\}$.



✓	✓	✓

4. For a.–c., if the function f is *injective*, circle it. → every y has at most 1 arrow

If the function f is not *injective* then in the blank spaces below that function, write down two elements a, b of the domain such that $a \neq b$ but $f(a) = f(b)$.

a. $f : [0, \infty) \to \mathbb{R}$ given by $f(x) = x^2 + 1$.

✓

__________ __________

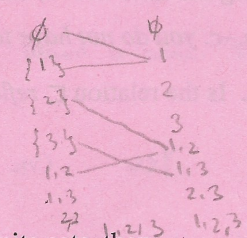
b. $f : \mathbb{Z} \to \mathbb{Z}$ given by $f(n) = \begin{cases} n & \text{if } n \text{ is even} \\ -n+1 & \text{if } n \text{ is odd.} \end{cases}$

✓

n=0 n=1 → $f(0) = 0$ $f(1) = 0$ $0 \neq 1$

c. $f : \mathcal{P}(\{1,2,3\}) \to \mathcal{P}(\{1,2,3\}) : S \mapsto S \cup \{1\}$.

✓



{2} {1,2}

$\{2\} \cup \{1\} = \{1,2\} = \{1,2\} \cup \{1\}$

5. State the *inclusion–exclusion principle*: If A and B are finite sets, then...

✓ $|A \cup B| = |A| + |B| - |A \cap B|$

↑ overlap

6. Let $S = \{0,1,2,3\} \times \{1,2,3\}$ and let $\sim$ be the relation on S defined by

$$(a,b) \sim (c,d) \quad \text{if} \quad ab = cd$$

trans
reflx
sym

List all the elements of the $\sim$-equivalence class $[(0,3)]$.

(0,1) (0,2) (0,3)
(1,1) (1,2) (1,3)
(2,1) (2,2) (2,3)
(3,1) (3,2) (3,3)

$\boxed{[(0,3)] = \{(0,1), (0,2), (0,3)\}}$

↑ anything whose product is 0 ↑ itself

✓	0	✓

7. One way to represent a relation is using a table.
Let $S = \{A, B, C, D\}$. We define a relation $\sqsubset$ on S by writing an X in the row for X and the column for Y in the table if (and only if) $X \sqsubset Y$:

	A	B	C	D
A	X	X		
B		X	X	
C			X	
D				X

For example, reading the first row: $A \sqsubset A$ and $A \sqsubset B$, but $A \not\sqsubset C$ and $A \not\sqsubset D$.
Using the table above, answer the following questions about the relation $\sqsubset$.
In a.–c. you do not have to justify "Yes."

a. Is the relation $\sqsubset$ *reflexive*? If you answered "No," why not?

Yes (b/c A⊏A, B⊏B, C⊏C, D⊏D)

b. Is the relation $\sqsubset$ *symmetric*? If you answered "No," why not?

NO symmetric means A⊏B and B⊏A are both relations
but looking at the diagram, B⊄A

c. Is the relation $\sqsubset$ *transitive*? If you answered "No," why not?

NO transitive means. if A⊏B and B⊏C, A⊏C
However, looking at the table A⊏B and B⊏C
are true, but A⊄C.

✓	✓	✓

8. Prove the following statement by *mathematical induction*: For all integers $n \geq 1$,

$$\sum_{k=1}^{n} k(k+1) = \frac{n(n+1)(n+2)}{3}$$

Try to keep your proof as organized as you possibly can!

Proof by induction $\quad P(n) = \sum_{k=1}^{n} k(k+1) = \frac{n(n+1)(n+2)}{3}$

① base case: $n=1$

LHS: $\sum_{k=1}^{1} k(k+1) = 1(1+1) = 2$ ✓ $\quad$ RHS: $\frac{1(1+1)(1+2)}{3} = \frac{1(2)(3)}{3} = 2$ ✓ $\quad$ LHS = RHS

② Inductive step. Assume $P(n)$ true for some fixed n. Want to show $P(n) \overset{\text{implies}}{\Longrightarrow} P(n+1)$

$$P(n+1) = \sum_{k=1}^{n+1} k(k+1) = \frac{(n+1)((n+1)+1)((n+1)+2)}{3}$$

LHS: $\sum_{k=1}^{n+1} k(k+1) = \underbrace{\sum_{k=1}^{n} k(k+1)}_{P(n)} + (n+1)((n+1)+1)$

$= \frac{n(n+1)(n+2)}{3} + (n+1)(n+2)$

$= \frac{n(n+1)(n+2) + 3(n+1)(n+2)}{3}$

$= \frac{(n^2+n)(n+2) + (3n+3)(n+2)}{3}$

$= \frac{n^3 + 2n^2 + n^2 + 2n + 3n^2 + 6n + 3n + 6}{3}$

$= \frac{n^3 + 6n^2 + 11n + 6}{3}$

$= \frac{(n+1)(n^2+5n+6)}{3}$

$= \frac{(n+1)(n+2)(n+3)}{3} = \text{RHS}$ ✓

RHS: $\frac{(n+1)(n+2)(n+3)}{3}$

$$\begin{array}{r} n^2 + 5n + 6 \\ n+1 \overline{\smash{)}\, n^3 + 6n^2 + 11n + 6} \\ n^3 + n^2 \\ \hline 5n^2 + 11n \\ 5n^2 + 5n \\ \hline 6n + 6 \end{array}$$

Therefore since $P(n) \Rightarrow P(n+1)$ $\forall n \geq 1$ $P(n)$ true