21F-MATH61-1 Midterm 2

DARREN ZHANG

TOTAL POINTS

40 / 40

QUESTION 1

1 ONOMATOPOEIA 10 / 10

✓ **- 0 pts** Correct

- 1 pts Part 1: Small mistake

- 5 pts Part 1: No credit

- 2.5 pts Part 2: Didn't order the other 8 letters

- 2.5 pts Part 2: Didn't allow M,N,P,T to be next to each other

- 3.5 pts Part 2: Counted arrangements where "MNPT" occurs as a substring

- 4 pts Part 2: Some attempt made, but no significant progress

- 5 pts Part 2: No credit

QUESTION 2

2 4 digit numbers 10 / 10

✓ **- 0 pts** Correct

- 4 pts In (1), did not make progress towards counting *strictly increasing* digits, or had approach that was not suited to this problem

- 2 pts In (2), did not remove the solutions that do not meet the constraint, or used an incorrect constraint.

- 2 pts Counted 2,3, and 4 digit numbers separately but did not include correct constraints, like to avoid counting the 2 digit numbers among the 3 and 4 digit numbers, for example.

- 4 pts No substantial progress towards a solution to (2)

- 3 pts Treated the selection of digits in (1) like independent events, and then multiplying possibilities

QUESTION 3

3 Recurrence relation 10 / 10

✓ **- 0 pts** Correct

- 2 pts Calculation error for auxiliary polynomial

- 2 pts Calculation error for initial condition (coefficients of solution)

- 4 pts Coefficients for auxiliary polynomial in wrong order

- 9 pts Did not compute auxiliary polynomial/find general solution

QUESTION 4

4 Genovia 10 / 10

✓ **- 0 pts** Correct

- 1 pts Minor issues with part 1

- 2 pts Issues with part 1.

- 3 pts Major issues with part 1

- 1 pts Minor issues with part 2.

- 1 pts Computed wrong pigeonhole in part 2.

- 2 pts Issues with pigeonhole

- 1 pts Minor issues with pigeonhole

- 1 pts Gave number of pigeonholes as number of pigeons in part 2.

- 3 pts Major issues with part 2

- 5 pts Part 2 blank

- 1 pts Minor issues with solution

- 10 pts Incorrect/blank

MATH 61 - Midterm 2

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Instructions:

- You are allowed a one page cheat sheet (front and back), but no other notes or resources.
- Show your work. Full points are only given for correct answers *with* adequate justification. A correct final answer with missing or substantially incorrect justification will not merit full points on a problem.
- You may write on the back of the problem sheet—this will be included in the scan and is preferable to showing additional work on scratch paper.
- You may leave your answer in combinatorial notation. For example, you could write $P(3,3)$, $\binom{6}{4}$, or $7!$ for answers instead of explicitly calculating the numbers.

Problem 1.

1. How many ways may the letters of the word ONOMATOPOEIA be arranged to form a distinct word?
   1 2 3 4 5 6 7 8 9 10 11 12

2. How many ways are there to arrange the letters of this word so that M, N, P, and T appear in alphabetical order (though not necessarily consecutively)? For example, they do appear in alphabetical order in

   OMONAPOTOEIA

   but not in ONOMATOPOEIA itself.

1) 12 letters
4 O's
2 A's

$$\frac{12!}{4!\,2!} \text{ ways}$$

This is because 12! ways to arrange 12 items, but the 4 O's are identical and the 2 A's are identical so we divide by 4! and 2!

2) Think of as cones and bones

[M] [N] [P] [T] are cones

The other 8 letters are bones.

\# of ways to arrange M, N, P, T in the letters is the # of solutions to

$$x_1 + x_2 + x_3 + x_4 + x_5 = 8$$

where x_1 = # of letters before M
x_2 = # of letters before N and after M
x_3 = # of letters before P and after N
x_4 = # of letters before T and after P
x_5 = # of letters after T

and $x_1, x_2, x_3, x_4 \geq 0$ and are integers

\# of solutions is $\binom{8+5-1}{5-1} = \binom{12}{4}$

\# of ways to arrange the remaining 8 letters OOAOOEIA in the 8 "bones" spots is $\frac{8!}{4!\,2!}$

total # ways where ONOMATOPOEIA can be arranged while M, N, P, T are in alphabetical order is $\binom{12}{4}\frac{8!}{4!\,2!}$

Problem 2.

1. How many 4 digit numbers are there with digits occurring in strictly increasing order? (Note: The first digit in a 4 digit number cannot be zero). left to right increasing
2. How many numbers of at most 4 digits are there whose digits sum to 10?

1)

can think of it as how many ways can we place 4 cones in 9 boxes

the value of the digit is the number of boxes to the left of the cone

so in the picture I drew, the digits are 1 2 5 7

In other words: ones digit $= x_1+x_2+x_3+x_4$, tens digit $= x_1+x_2+x_3$, 100s digit: x_1+x_2, 1000s digit $= x_1$

equal to the # of solutions to

$x_1+x_2+x_3+x_4+x_5=9$

$x_1\geq 1$, $x_2\geq 1$, $x_3\geq 1$, $x_4\geq 1$

equivalent to

$x_1'+x_2'+x_3'+x_4'+x_5'=5$

no constraint

# of solutions is $\binom{5+5-1}{5-1}=\binom{9}{4}$

There are $\binom{9}{4}$ 4-digit numbers with digits occurring in strictly increasing order from left to right

further explanation, $x_1\geq 1$ because the first digit can't be 0

$x_2\geq 1$ because digit in 100th place needs to be at least 1 greater than digit in 1000th place

$x_3\geq 1$ because digit in 10's place needs to be at least 1 greater than digit in 100th place

$x_4\geq 1$ because digit in ones place needs to be at least 1 greater than digit in 10's place

x_5 has no constraint because ones digit can equal 9 which is when $x_5=0$

2) $x_1+x_2+x_3+x_4=10$ where x_1 is 1000th digit, x_2 is 100th digit, x_3 is tens digit, x_4 is ones digit

$x_1\leq 9$, $x_2\leq 9$, $x_3\leq 9$, $x_4\leq 9$

# of solutions = # of solutions with no constraint − # of solutions with $x_1\geq 10$ − # of solutions with $x_2\geq 10$ − # of solutions with $x_3\geq 10$ − # of solutions with $x_4\geq 10$

don't need to worry about overlapping of two or more constraints since solutions with $x_1\geq 10$, solutions with $x_2\geq 10$, solutions with $x_3\geq 10$, and solutions with $x_4\geq 10$ are mutually exclusive. This is because if one number is 10 or larger, there is no way for another number to be 10 or larger and still have the sum of all 4 numbers equal 10

① # of solutions with no constraint

$x_1+x_2+x_3+x_4=10$

$\binom{10+4-1}{4-1}=\binom{13}{3}$

② # of solutions with $x_1\geq 10$

$x_1+x_2+x_3+x_4=10$

$x_1\geq 10$

equivalent to

$x_1'+x_2+x_3+x_4=0$

no constraint

$\binom{0+4-1}{4-1}=\binom{3}{3}=1$

③ # of sol. with $x_2\geq 10$

$x_1+x_2+x_3+x_4=10$

$x_2\geq 10$

equivalent to:

$x_1+x_2'+x_3+x_4=0$

no constraint

$\binom{0+4-1}{4-1}=\binom{3}{3}=1$

④ # of sol. with $x_3\geq 10$

$x_1+x_2+x_3+x_4=10$

$x_3\geq 10$

equivalent to:

$x_1+x_2+x_3'+x_4=0$

no constraint

$\binom{0+4-1}{4-1}=\binom{3}{3}=1$

continue on back

⑤ # of solutions with $x_4 \geq 10$

$$x_1 + x_2 + x_3 + x_4 = 10$$
$$x_4 \geq 10$$

equivalent to

$$y_1 + y_2 + x_3 + x_4' = 0$$

no constraint

$$\binom{0+4-1}{4-1} = \binom{3}{3} = 1$$

# of solutions to

$$x_1 + x_2 + x_3 + x_4 = 10$$
$$x_1 \leq 9,\ x_2 \leq 9,\ x_3 \leq 9,\ x_4 \leq 9$$

is

$$① - ② - ③ - ④ - ⑤ = \binom{13}{3} - 1 - 1 - 1 - 1 = \binom{13}{3} - 4$$

the amount of numbers of at most 4 digits whose digits sum up to 10 is $\binom{13}{3} - 4$

Problem 3. Suppose the sequence $\{s_n\}_{n=0}^{\infty}$ satisfies the equation

$$3s_n - 21s_{n-1} = -30s_{n-2}$$

for all $n \geq 2$ with $s_0 = 5$ and $s_1 = 19$. Find s_n. Note: Your answer should be an explicit formula for s_n in terms of n.

$3s_n - 21s_{n-1} = -30s_{n-2}$

$3s_n = 21s_{n-1} - 30s_{n-2}$

$s_n = 7s_{n-1} - 10s_{n-2}$ (this is a 2nd order homogeneous linear recurrence relation)

auxiliary polynomial:

$t^2 - 7t - (-10)$

$t^2 - 7t + 10$

Solve for roots

$t^2 - 7t + 10 = 0$

$(t-5)(t-2) = 0$

$t = 5, 2$

$s_n = \alpha(5)^n + \beta(2)^n$

$s_0 = 5 = \alpha(5)^0 + \beta(2)^0$

$s_1 = 19 = \alpha(5)^1 + \beta(2)^1$

$\Rightarrow$ $5 = \alpha + \beta$, $19 = 5\alpha + 2\beta$ $\qquad \alpha = 5 - \beta$

$19 = 5(5-\beta) + 2\beta$

$19 = 25 - 5\beta + 2\beta$

$-6 = -3\beta$

$\beta = 2$

$\alpha = 5 - \beta = 5 - 2 = 3$

$\alpha = 3, \beta = 2$

$s_n = 3(5)^n + 2(2)^n$

$\boxed{s_n = 3(5)^n + 2^{n+1}}$

Check answer:

$s_2 = 7s_1 - 10s_0 = 7(19) - 10(5) = 133 - 50 = 83$

$s_2 = 3(5)^2 + 2^{2+1} = 3(25) + 8 = 75 + 8 = 83$

366 possible birthdays b/c leap years

Problem 4. In the small European country of Genovia, the population is 2 million. Assume that every resident of Genovia is born in one of Genovia's 10 cities and studies at one of Genovia's 5 universities.

1. Show that there are at least 100 residents of Genovia who all have their home town, university, and birthday in common.

2. How many residents of Genovia can you be certain share a hometown and birthday in common?

1) 10 cities = 10 hometowns
5 universities
366 possibly birthdays

$10 \cdot 5 \cdot 366$ possibly hometown, university, and birthday combinations

$366 \times 50 = 18300$

2 million residents are the "pigeons"
18,300 hometown, university, birthday combos are the "pigeonholes"

so by the pigeonhole principle
at least $\left\lceil \frac{2,000,000}{18,300} \right\rceil$ people have the same hometown, university, and birthday

$\left\lceil \frac{2,000,000}{18,300} \right\rceil = \left\lceil 100 + \frac{170,000}{18,300} \right\rceil \geq 101$ since this value is greater than 100, there are at least 100 residents of Genovia who all have their hometown, university, and birthday in common.

2) pigeons is the residents of Genovia
pigeonhole is a hometown birthday combination

2,000,000 residents
$10 \cdot 366 = 3,660$ hometown bday combos

I can be certain at least $\left\lceil \frac{2,000,000}{3,660} \right\rceil$ residents of Genovia share a hometown and birthday in common.