61 Midterm 2

Nathan George Midkiff

TOTAL POINTS

59 / 70

QUESTION 1

1 Exercise 1 **10 / 10**

- **- 2 pts** (1) Incorrect [correct response T]
- **- 2 pts** (2) Incorrect [correct response F]
- **- 2 pts** (3) Incorrect [correct response T]
- **- 2 pts** (4) Incorrect [correct response T]
- **- 2 pts** (5) Incorrect [correct response F]
- ✓ **- 0 pts All correct**

QUESTION 2

2 Exercise 2 **9 / 10**

- **- 5 pts** (1) Incorrect
- **- 1 pts** (2) Incorrect: off-by-one error.
- **- 2 pts** (2) Incorrect: gave answer m choose n instead of (m choose n) * n!.
- ✓ **- 5 pts (2) Incorrect: other reason**
- **- 0 pts** Both correct
- **- 10 pts** Skipped
- **+ 4 Point adjustment**

(2) The product is correct but equal to m!/(m - n)!.

QUESTION 3

3 Exercise 3 **0 / 10**

- **- 0 pts** Correct
- **- 3 pts** Miscounted letters/repeats (e.g. answered 10!/2!)
- ✓ **- 10 pts Not graded**
- **- 8 pts** Only recognized 10! permutations without considering repeats. (Or dealt with repeats incorrectly.)

QUESTION 4

4 Exercise 4 **10 / 10**

- ✓ **- 0 pts Correct**
- **- 10 pts** Skipped
- **- 7 pts** No further than expanding binomials (correctly)
- **- 2 pts** Small algebra error
- **- 4 pts** Made progress, but substantial algebra error or didn't finish
- **- 8 pts** Showed some indication of what the individual terms mean combinatorially, but not how they're related
- **- 0 pts** Click here to replace this description.

Very clean!

QUESTION 5

5 Exercise 5 **10 / 10**

- ✓ **- 0 pts Correct**
- **- 10 pts** Incorrect
- **- 3 pts** Apply Pigeonhole Incorrectly
- **- 8 pts** Didn't apply Pigeonhole
- **- 10 pts** Not Graded
- **- 5 pts** Flawed argument, did not consider birthdays on same day

QUESTION 6

6 Exercise 6 **10 / 10**

- ✓ **- 0 pts Correct**
- **- 10 pts** skipped
- **- 3 pts** Algebra mistakes in solving for coefficients
- **- 5 pts** Incorrect auxiliary polynomial
- **- 9 pts** Incorrect, no attempt at application of method
- **- 6 pts** No solution after finding aux polynomial and root
- **- 5 pts** Incorrect solution form
- **- 5 pts** Incorrect root for aux poly

QUESTION 7

7 Exercise 7 **10 / 10**

✓ **- 0 pts** **Correct**

- 10 pts Skipped

- 1 pts Correct answer, but only justified by a picture

- 1 pts Correct answer, but missing justification

- 4 pts Omitted empty graph

- 9 pts Incorrect, with no clear justification

- 8 pts Miscounted because had edges not connected to the given vertices

- 8 pts Miscounted because counted vertices that were connected to other vertices

- 5 pts Did not include the empty graph in the count

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MATH 61 - MIDTERM EXAM 1

0.1. **Instructions.** This is a 50 minute exam. You should feel free to quote any theorems proved in class, as well as anything proved in the homework or discussion section. There are 7 questions—on the real exam, you are required to do the first true/false question, and choose 5 of the remaining 6. Only 5 problems other than the true/false question will be graded so *you should indicate which problems you want graded by marking the one you do not want graded with an X*, in the case that you attempt all 6. Each question is worth 10 points. Unless otherwise specified, you are required to justify your answers.

Exercise 0.1. Indicate whether the following statements are true or false. You do not need to justify your answer.

(1) Suppose X and Y are finite sets with $|X|$ and $|Y|$ even. Then $|X \Delta Y|$ is even.

(2) If $0 \leq k \leq n$,

$$\binom{n}{k} = \binom{n-k}{k}.$$

(3) If $|X| = n$, there are n^n functions from X to X.

(4) There are $2^n - 1$ sequences of 0s and 1s of length n with at least one 0.

(5) Any sequence $\{a_n\}$ satisfying the recurrence relation

$$a_{n+2} = 4a_{n+1} + 2a_n$$

must have only even terms.

1) True

2) $\frac{n!}{(n-k)!\,k!} = \frac{(n-k)!}{(n-2k)!\,k!}$ False

3) True

4) True

5) False

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Exercise 0.2. Suppose X and Y are finite sets with $n = |X| < |Y| = m$.

(1) Show there are $n!$ functions from X to X that are both injective and surjective.

(2) How many injective functions are there from X to Y?

1) If $X = \{x_1, \ldots, x_n\}$, then you can choose n values that could be $f(x_1)$. Then, because the function is injective, there are $n-1$ values that could be $f(x_2)$. Each element has one less possibility for its value until $f(x_n)$ has only one possible value remaining. Because each element in the domain must go to a unique element in the codomain, and the size of the domain = the size of the codomain, there is some element in the domain that goes to each element in the codomain, so all of the functions are surjective. So there are $n(n-1) \cdot \ldots \cdot 1 = n!$ such functions.

2) $m \cdot (m-1) \cdot (m-2) \cdot \ldots \cdot (m-n+1) = \dfrac{m!}{n!}$

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Do Not Grade

Exercise 0.3. How many words can be obtained by rearranging the letters in

CALIFORNIA

(**Note:** they do not need to be real words and you can leave your answer in combinatorial notation)?

C - 1 A - 2 L - 1 I - 2 F - 1 O - 1 R - 1 N - 1

$\binom{10}{2} \cdot \binom{8}{2} \cdot 6!$

$\frac{10! \cdot \cancel{8!} \cdot \cancel{6!}}{\cancel{8!}\, 2!\, \cancel{6!}\, 2!} = \boxed{\frac{10!}{4}}$

AABC
ABAC
BAAC
AACB
ABCA
BACA
ACAB
ACBA
BCAA
CAAB
CABA
CBAA

$\binom{4}{2} \cdot 2! = \frac{4!}{2} = 2 \cdot 3 \cdot 2 = 12$

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Exercise 0.4. Show that if $2 \leq k \leq n$, then

$$\binom{n}{k} + 2\binom{n}{k-1} + \binom{n}{k-2} = \binom{n+2}{k}.$$

If you have $n+2$ elements and you need to choose k of them, there are $\binom{n+2}{k}$ ways to do it. Also, if you mark two of the elements as unique, there are 4 cases as to how to pick k elements.

Case 1: You don't pick either of the unique elements, So now you have n elements left to pick k elements, which you can do in $\binom{n}{k}$ ways.

Case 2: You pick unique element #1 but not #2. So now you have n elements left and you still need to pick $k-1$ elements, which you can do in $\binom{n}{k-1}$ ways.

Case 3: You pick unique element #2 but not #1. For the same reason as case 2, you can do this in $\binom{n}{k-1}$ ways.

Case 4: You pick both unique elements. Now you have n elements left and you need to pick $k-2$, which you can do in $\binom{n}{k-2}$ ways.

No two cases intersect, so the total # ways to pick k elements from a set of $n+2$ elements is $\binom{n}{k} + \binom{n}{k-1} + \binom{n}{k-1} + \binom{n}{k-2} = \binom{n}{k} + 2\binom{n}{k-1} + \binom{n}{k-2}$.

So therefore $\binom{n}{k} + 2\binom{n}{k-1} + \binom{n}{k-2} = \binom{n+2}{k}$

$\begin{array}{r} 185 \\ \underline{185} \\ 370 \end{array}$

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Exercise 0.5. Let's pretend that there are no leap years so every year has 365 days. Show that if there are 185 students in our class, then there are two students who have either the same birthday or have consecutive birthdays.

let $\{b_1, b_2, \dots, b_{185}\}$ be the set containing each student's birthday.

$\{b_1+1, b_2+1, \dots, b_{185}+1\}$ is the set of proceeding birthdays. Consider the union of the two sets, $\{b_1, \dots, b_{185}, b_1+1, \dots, b_{185}+1\}$. This set contains $2 \cdot 185 = 370$ elements, but the possible values for any element in the set is $\{1, 2, \dots 366\}$. Therefore, by the pigeonhole principle, at least two elements in the set have the same value, so either $b_i = b_j$ for $i \neq j$, or $b_i = b_j + 1$, or $b_i + 1 = b_j + 1 \Rightarrow b_i = b_j$. Therefore at least two students have the same or consecutive birthdays.

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Exercise 0.6. Find the solution to the recurrence relation

$$a_n = 6a_{n-1} - 9a_{n-2},$$

subject to the initial conditions $a_0 = 2$ and $a_1 = 9$.

$a_n - 6a_{n-1} + 9a_{n-2} = 0$

$k^2 - 6k + 9 = 0 \Rightarrow (k-3)(k-3) = 0 \quad k = 3$

$a_n = C_1 \cdot 3^n + C_2 \cdot n \cdot 3^n$

$a_0 = C_1 \cdot 3^0 + C_2 \cdot 0 \cdot 3^0 = C_1 = 2$

$a_1 = 3c_1 + 3c_2 = 9$

$\Rightarrow 6 + 3c_2 = 9$

$\Rightarrow c_2 = 1$

$\boxed{a_n = 2 \cdot 3^n + n \cdot 3^n}$

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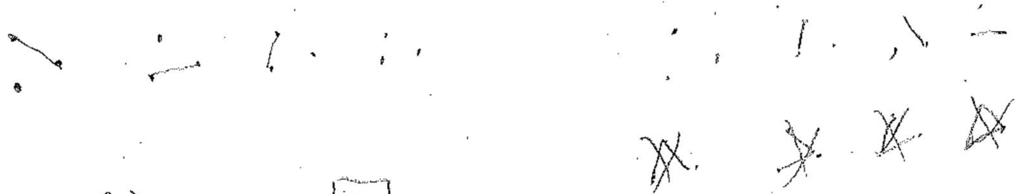
Exercise 0.7. Suppose $G = (V, E)$ is a graph. Say that a vertex $v \in V$ is *unfriendly* if it is connected by an edge to no other vertex. If $|V| = 3$, how many possible graphs are there with vertex set V that contain an unfriendly vertex?



$3 \cdot \binom{2}{2} + 1 \cdot 1 = \boxed{4}$