21F-MATH61-1 Midterm 1

DARREN ZHANG

TOTAL POINTS

40 / 40

QUESTION 1

1 Induction Question 10 / 10

✓ **- 0 pts** **Correct**

- 1 pts Minor issues with induction step

- 2 pts Started working from what was supposed to be proven.

- 3 pts Issues with inductive step

- 5 pts Major issues with induction step

- 7 pts Induction step incorrect

- 8 pts Some good ideas, but not a solution

QUESTION 2

2 Sequence Length Question 10 / 10

✓ **- 0 pts** **Correct**

- 4 pts 1 incorrect

- 4 pts 2 incorrect

QUESTION 3

3 Difference of Relations Question 10 / 10

✓ **- 0 pts** **Correct**

- 2 pts Unclear explanation for R - S being symmetric

- 2 pts Unclear explanation for R - S being irreflexive

- 8 pts Solved the problem for a specific example rather than the general statement.

- 3 pts Showed that R-S is not an equivalence relation for a special case of R and S, but did not prove it for all R and S.

- 4 pts No work for 2.

- 10 pts Blank

QUESTION 4

4 Product of Functions Question 10 / 10

✓ **- 0 pts** **Correct**

- 2.5 pts Part 1: Vague argument about "unique values" / "distinct outputs" / etc.

- 3.5 pts Part 1: Argued by cardinality inequality

- 3.5 pts Part 1: "Proof" by example

- 4 pts Part 1: Partial credit for correct recall of the definition of injective

- 5 pts Part 1: No Credit / Nothing Written

- 1 pts Part 2: Correct counterexample, but not fully specified

- 1 pts Part 2: Correct counterexample, but no/incorrect proof that h is not surjective

- 2.5 pts Part 2: Correct idea, but no correct specific counterexample given

- 4 pts Part 2: Partial credit for correct recall of the definition of surjective

- 5 pts Part 2: No Credit / Nothing Written

- 10 pts No Credit / Nothing Written

Excellent

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[redacted]

Instructions:

- You are allowed a one page cheat sheet (front and back), but no other notes or resources.
- Show your work. Full points are only given for correct answers *with* adequate justification. A correct final answer with missing or substantially incorrect justification will not merit full points on a problem.

Problem 1. Prove by induction that

$$1! \cdot 1 + 2! \cdot 2 + \ldots + n! \cdot n = (n+1)! - 1$$

for all $n \geq 1$.

Base case: $n=1$

$$1! \cdot 1 = (1+1)! - 1$$
$$1 \cdot 1 = 2! - 1$$
$$1 = 2 \cdot 1 - 1$$
$$1 = 2 - 1$$
$$1 = 1$$

LHS = RHS ✓ Base case is true

Inductive hypothesis:
Assume for some $n \in \mathbb{N}$ that $1! \cdot 1 + 2! \cdot 2 + \ldots + n! \cdot n = (n+1)! - 1$

Inductive step

$$1! \cdot 1 + 2! \cdot 2 + \ldots + n! \cdot n + (n+1)!(n+1) = (1! \cdot 1 + 2! \cdot 2 + \ldots + n! \cdot n) + (n+1)!(n+1)$$

by inductive hypothesis

$$1! \cdot 1 + 2! \cdot 2 + \ldots + n! \cdot n + (n+1)!(n+1) = (n+1)! - 1 + (n+1)!(n+1)$$
$$= 1 \cdot (n+1)! + (n+1)(n+1)! - 1$$
$$= (1 + n + 1)(n+1)! - 1$$
$$= (n+2)(n+1)! - 1$$
$$= (n+2)! - 1$$
$$= ((n+1)+1)! - 1$$

So the $n+1$ case which says $1! \cdot 1 + 2! \cdot 2 + \ldots + n! \cdot n + (n+1)!(n+1) = ((n+1)+1)! - 1$ is true.

Therefore, by induction

$1! \cdot 1 + 2! \cdot 2 + \ldots + n! \cdot n = (n+1)! - 1$ is true for all $n \geq 1$

Problem 2. Let $\mathcal{S}$ be the set of all finite strings over $\{0, 1\}$. Let $l : \mathcal{S} \to \mathbb{Z}^{\geq 0}$ be the function defined so that $l(s)$ is the length of s, for all strings $s \in \mathcal{S}$ (where $\mathbb{Z}^{\geq 0}$ denotes the set of non-negative integers).

1. Is l injective?

2. Is l surjective?

Justify your answers.

1. No, ℓ is not injective. If two strings have the same length, they don't necessarily have to be the same string.
for example "01" and "10" both have length 2, but they are not the same string.

2. Yes, ℓ is surjective because for any $n \in \mathbb{Z}^{\geq 0}$ there exists a string over $\{0,1\}$ of length n.

Base Case: When $\ell(s) = 0$, s is the empty string.

Assume for some $n \in \mathbb{Z}^{\geq 0}$ we have s where $\ell(s) = n$.

Then if we concatenate either a "1" or a "0" to the end of s, we will have a string of length $n+1$.
Thus, for any $n \in \mathbb{Z}^{\geq 0}$, there exists a string where its length is equal to n

Problem 3. Suppose X is a non-empty set and R and S are relations on X.

1. Show that if R and S are symmetric and reflexive, then $(R - S)$ is a symmetric and irreflexive relation on X.

2. Show that if R and S are equivalence relations, then $R - S$ is not an equivalence relation.

1. By definition, since R and S are reflexive, $\forall x \in X$ $(x,x) \in R$ and $(x,x) \in S$

Thus, the $\{(x,x) \mid x \in X\} \subseteq R \cap S$

Therefore, R-S is irreflexive. This is because:

$R-S = R-(R \cap S)$

We know $\{(x,x) \mid x \in X\} \subseteq R \cap S$ so R-S doesn't contain (x,x) for any $x \in X$

R-S is symmetric because by definition of symmetric $\forall (x,y) \in S$, $(y,x) \in S$ and $\forall (x,y) \in R$, $(y,x) \in R$

Therefore, if $(x,y) \in R$ and S, then $(y,x) \in R$ and S. Thus, $\forall (x,y) \in R \cap S$, $(y,x) \in R \cap S$

Thus, if $(x,y) \in R$ but $(x,y) \notin R \cap S$, then $(y,x) \in R$ and $(y,x) \notin R \cap S$

Since $R-S = R-(R \cap S)$, $\forall (x,y) \in R$ and $(x,y) \notin R \cap S$, $(y,x) \in R$ and $(y,x) \notin R \cap S$

So $\forall (x,y) \in (R-S)\ \exists (y,x) \in (R-S)$

and thus R-S is symmetric

2. If R and S are equivalence relations, R and S are both reflexive.

As shown in part 1, if R and S are reflexive, R-S is irreflexive

If R-S is irreflexive, it can't be reflexive unless X is the empty set. Since R-S is not reflexive, R-S is not an equivalence relation.

Note, the case where X is the empty set is trivial because R and S would both be empty and R-S would also be empty.

Problem 4. Suppose $f : X \to Y$ and $g : X \to Z$ are functions. Define a function $h : X \to Y \times Z$ by

$$h(x) = (f(x), g(x))$$

for all $x \in X$.

1. Show that if f and g are injective, then h is injective.
2. Show, by example, that even if $f : X \to Y$ and $g : X \to Z$ are surjective, h is not necessarily surjective.

1) By definition of injective, $\forall x_1, x_2 \in X$, if $f(x_1) = f(x_2)$, then $x_1 = x_2$

and $\forall x_1, x_2 \in X$, if $g(x_1) = g(x_2)$, then $x_1 = x_2$

Thus, $\forall x_1, x_2 \in X$, if $f(x_1) = f(x_2)$ and $g(x_1) = g(x_2)$, then $x_1 = x_2$

and thus, $\forall x_1, x_2 \in X$, if $h(x_1) = (f(x_1), g(x_1)) = h(x_2) = (f(x_2), g(x_2))$

then $x_1 = x_2$

and thus h is injective if f and g are injective

2) for example: $X = \{1,2,3\}$, $Y = \{1,2\}$, $Z = \{1,2,3\}$

$f = \{(1,2),(2,2),(3,1)\}$ surjective ✓ b/c $Y = \{1,2\}$

$g = \{(1,1),(2,2),(3,3)\}$ surjective ✓ b/c $Z = \{1,2,3\}$

Then $h = \{(1,(2,1)), (2,(2,2)), (3,(1,3))\}$

and h is not surjective as the codomain of h is $Y \times Z = \{(1,1),(1,2),(1,3),(2,$ [cut off] $(2,2),(2,3)\}$

and the range of h does not contain $(1,1)$, $(1,2)$, and $(2,3)$