Problem 1. Prove by induction that

$$1! \cdot 1 + 2! \cdot 2 + \ldots + n! \cdot n = (n+1)! - 1$$

for all $n \geq 1$.

Proof. The base case is $n = 1$; the left-hand side is $1! \cdot 1 = 1$ and the right-hand side is $2! - 1 = 1$.
Assume for induction that it has been established for some arbitrary k that

$$1! \cdot 1 + 2! \cdot 2 + \ldots + k! \cdot k = (k+1)! - 1.$$

Then we calculate

$$\begin{aligned} 1! \cdot 1 + 2! \cdot 2 + \ldots + k! \cdot k + (k+1)! \cdot (k+1) &= (k+1)! - 1 + (k+1)! \cdot (k+1) \\ &= (k+1)!(1 + (k+1)) - 1 \\ &= (k+2)! - 1. \end{aligned}$$

This completes the induction step, and the proof. □

Problem 2. Let $\mathcal{S}$ be the set of all finite strings over $\{0, 1\}$. Let $l : \mathcal{S} \to \mathbb{Z}^{\geq 0}$ be the function defined so that $l(s)$ is the length of s, for all strings $s \in \mathcal{S}$ (where $\mathbb{Z}^{\geq 0}$ denotes the set of non-negative integers).

1. Is l injective?
2. Is l surjective?

Justify your answers.

Proof. The function l is not injective since, for example, 0 and 1 are both length 1 strings which are distinct. The function l is surjective since, for any $n \in \mathbb{Z}^{\geq 0}$, the string $0 \ldots 0$ of length n consisting of all 0s, for example, has length n, so $l(0 \ldots 0) = n$. □

Problem 3. Suppose X is a non-empty set and R and S are relations on the non-empty X.

1. Show that if R and S are symmetric and reflexive, then $(R - S)$ is a symmetric and irreflexive relation on X.

2. Show that if R and S are equivalence relations, then $R - S$ is not an equivalence relation.

Proof. (1). First we show $R - S$ is symmetric. If $(x, y) \in (R - S)$, then $(x, y) \in R$ and $(x, y) \notin S$. As R and S are symmetric, it follows that $(y, x) \in R$ and $(y, x) \notin S$. This shows $(y, x) \in (R - S)$, and therefore $R - S$ is symmetric. Next we show $R - S$ is irreflexive. Actually this only requires that S be reflexive. Since for all $x \in X$, $(x, x) \in S$, we know $(x, x) \notin R - S$ which means $R - S$ is irreflexive.
(2). Follows from (1) since $R - S$ is irreflexive (and therefore not reflexive, since X is non-empty). $\square$

Problem 4. Suppose $f : X \to Y$ and $g : X \to Z$ are functions. Define a function $h : X \to Y \times Z$ by

$$h(x) = (f(x), g(x))$$

for all $x \in X$.

1. Show that if f and g are injective, then h is injective.
2. Show, by example, that even if $f : X \to Y$ and $g : X \to Z$ are surjective, h is not necessarily surjective.

Proof. (1). Suppose $x \neq x'$. Then $f(x) \neq f(x')$ and therefore $(f(x), g(x)) \neq (f(x'), g(x'))$ which shows $h(x) \neq h(x')$. Therefore h is injective.
(2). Let $X = Y = Z = \mathbb{R}$ and take $f(x) = x$ and $g(x) = x$. Then f and g are clearly surjective with $h(x) = (x, x)$ for all $x \in \mathbb{R}$. It follows that $(1, 2) \in \mathbb{R} \times \mathbb{R}$ is not in the range, so h is not surjective. $\square$