Math 61 Midterm 1

NICHOLAS DEAN

TOTAL POINTS

11 / 13

QUESTION 1

1 Induction Question 2 / 4

- **- 0 pts** Correct
- ✓ **- 1.5 pts Base case was trivialized**
- **- 0.75 pts** Base case and/or induction step argument does not explain both inclusions between a pair of sets.
- ✓ **- 1 pts Induction step carried out incorrectly in form (obscuring the role of induction in the proof)**
- **- 1 pts** Induction step carried out incorrectly in content
- **- 0.5 pts** Misunderstanding of union
- **- 0.5 pts** Unclear logic in base case
- **- 0.5 pts** Handles arbitrary elements or sets incorrectly
- **- 0.5 pts** Misunderstanding or unclear use of equality and/or implication
- **- 0.5 pts** Misunderstanding of set builder notation or sets and their cardinalities
- **- 0.25 pts** Minor unpacking error
- **- 0.25 pts** Misuse of notation
- **- 1 pts** Misunderstanding of cartesian product
- **+ 0.5 Point adjustment**

QUESTION 2

2 Relation Question 4 / 4

Part (a): (i) $$R$$ can be both anti-symmetric and symmetric simultaneously, or (ii) $$R$$ can be not(anti-symmetric) and not(symmetric) simultaneously.

- ✓ **- 0 pts Correct example: gave a relation which was both symmetric and anti-symmetric, or a relation which was neither.**
- **- 1 pts** Unclear or imprecise mathematical statements made. For example, the argument did not give a clear explanation of both properties, or gave some correct examples (of a relation being both or neither properties), but also some incorrect examples.
- **- 2 pts** Missing or incorrect example, or major misunderstandings

Part (b): $$ R $$ can be not(anti-reflexive) and not(reflexive) simultaneously

- ✓ **- 0 pts Correct: gave an example of a relation which was neither reflexive nor anti-reflexive. (Or, gave the example of $$ X = \emptyset $$ and $$ R = \emptyset $$)**
- **- 1 pts** Unclear or imprecise mathematical statements made. For example, the argument did not give a clear explanation of both properties, or gave some correct examples, but also some incorrect examples.
- **- 2 pts** Missing or incorrect example, or major misunderstandings

QUESTION 3

3 Function Question 3 / 3

- ✓ **- 0 pts Correct**
- **- 1 pts** incomplete or incorrect argument for injectivity when n = 1
- **- 1 pts** incomplete or incorrect argument for surjectivity
- **- 1 pts** incomplete or incorrect argument for non-injectivity for n >1

QUESTION 4

4 Counting Question 2 / 2

- ✓ **- 0 pts Correct**
- **- 0.5 pts** 16! ways with Averie first and 16! ways with Charlie last
- **- 0.5 pts** 15! ways with Averie first and Charlie last

- 1 pts 2(16!)-15! total by Inclusion-Exclusion Principle

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Midterm 1 **MATH 61**

Instructions:

- You have from Friday 23 October 2020 at 00.00am to 11.59pm Pacific Time to solve this exam.
- Scan your solutions and upload them to Gradescope by Friday 23 October at 11.59pm Pacific Time. You should submit readable scans, and not pictures of your solutions. Please make sure to match the problems on the exam template with the respective parts in your solutions.
- This exam is open book, and you are allowed to use the textbook, and all resources from the lecture, or similar resources.
- You are not allowed to ask for help from others, nor give help to others taking this exam. Students suspected of academic dishonesty may be reported to the Dean of Students.

Code of honour

Academic integrity is of the uttermost importance. By taking part in this evaluation, you are accepting the following code of honor:

I certify on my honor that I have neither given nor received any help, or used any non-permitted resources, while completing this evaluation.

Problem 1. Let $n \geq 2$ be a natural number. Let $A_1, \ldots, A_n$ and C be arbitrary sets. Using mathematical induction, show that

$$\left(\bigcup_{i=1}^{n} A_i\right) \times C = \bigcup_{i=1}^{n} (A_i \times C) .$$

Let $P(n)$ be "$\left(\bigcup_{i=1}^{n} A_i\right) \times C = \bigcup_{i=1}^{n} (A_i \times C)$"

Base case: $n = 2$

$$\left(\bigcup_{i=1}^{3} A_i\right) \times C = \bigcup_{i=1}^{2} (A_i \times C)$$

$$(A_1 \cup A_2) \times C = (A_1 \times C) \cup (A_2 \times C)$$

apply the Distributive Law to the left hand side

$$(A_1 \times C) \cup (A_2 \times C) = (A_1 \times C) \cup (A_2 \times C)$$

left-hand side = right-hand side

base case is satisfied, and thus, $P(2)$ is true.

Induction step: Let's assume that $P(n)$ is true for some positive integer n. Need to prove $P(n+1)$:

$$\left(\bigcup_{i=1}^{n+1} A_i\right) \times C = \bigcup_{i=1}^{n+1} (A_i \times C)$$

$$(A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_{n+1}) \times C = (A_1 \times C) \cup (A_2 \times C) \cup (A_3 \times C) \cup \ldots \cup (A_{n+1} \times C)$$

similar to before, apply Distributive Law to LHS

$$(A_1 \times C) \cup (A_2 \times C) \cup (A_3 \times C) \cup \ldots \cup (A_{n+1} \times C) = (A_1 \times C) \cup (A_2 \times C) \cup (A_3 \times C) \cup \ldots \cup (A_{n+1} \times C)$$

left-hand side = right-hand side

Thus, $P(n+1)$ is true.

By induction, we can conclude that $P(n)$ is true for $n \geq 2$.

1 Induction Question 2 / 4

- **- 0 pts** Correct
- ✓ **- 1.5 pts** **Base case was trivialized**
- **- 0.75 pts** Base case and/or induction step argument does not explain both inclusions between a pair of sets.
- ✓ **- 1 pts** **Induction step carried out incorrectly in form (obscuring the role of induction in the proof)**
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- **- 1 pts** Misunderstanding of cartesian product
- **+ 0.5** **Point adjustment**

Problem 2. Let R be a relation on a set X.

(a) Explain in words why the statement "R is anti-symmetric" is not the negation of the statement "R is symmetric". Provide examples to illustrate your explanation.

(b) Explain in words why the statement "R is anti-reflexive" is not the negation of the statement "R is reflexive". Provide examples to illustrate your explanation.

a.) Let's first look at the definitions of symmetric and anti-symmetric. A relation R is symmetric if whenever $(a,b) \in R$, then $(b,a) \in R$. A relation R is anti-symmetric if both $(a,b) \in R$ and $(b,a) \in R$, then $a=b$. The reason "R is anti-symmetric" is not the negation of the statement "R is symmetric" is because a relation R can be both symmetric and anti-symmetric. If the statements were negations of one another, this would not be possible as one must be true when the other is false. See examples below:

both symmetric and anti-symmetric:

$R_1 = \{(0,0),(1,1),(2,2),(3,3)\}$

- symmetric b.c. $(b,a) \in R$ when $(a,b) \in R$
- anti-symmetric b.c. when (a,b) and $(b,a) \in R$, then $a=b$

neither symmetric nor anti-symmetric:

$R_2 = \{(1,2),(2,1),(3,4)\}$

- not symmetric b.c. $(4,3) \notin R$
- not anti-symmetric b.c. $(1,2) \in R$ and $(2,1) \in R$

b.) Let's first look at the definitions of reflexive and anti-reflexive. A relation R is reflexive where for every $x \in X$, then $(x,x) \in R$. A relation R is irreflexive if for every $x \in X$, then $(x,x) \notin R$. The reason "R is anti-reflexive" is not the negation of the statement "R is reflexive" because a relation R can be neither anti-reflexive nor reflexive.

* continued on back

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If the statements were negations of one another, this would not be possible as one must be true when the other is false. See examples below:

neither anti-reflexive nor reflexive:

$$R = \{(1,1), (1,2), (2,1)\}$$ on the set of integers $\{1,2\}$

- not reflexive b.c. $(2,2)$ is not in R
- not anti-reflexive b.c. $(1,1)$ is in R

2 Relation Question **4 / 4**

Part (a): (i) $$R$$ can be both anti-symmetric and symmetric simultaneously, or (ii) $$R$$ can be not(anti-symmetric) and not(symmetric) simultaneously.

✓ **- 0 pts** **Correct example: gave a relation which was both symmetric and anti-symmetric, or a relation which was neither.**

- 1 pts Unclear or imprecise mathematical statements made. For example, the argument did not give a clear explanation of both properties, or gave some correct examples (of a relation being both or neither properties), but also some incorrect examples.

- 2 pts Missing or incorrect example, or major misunderstandings

Part (b): $$ R $$ can be not(anti-reflexive) and not(reflexive) simultaneously

✓ **- 0 pts** **Correct: gave an example of a relation which was neither reflexive nor anti-reflexive. (Or, gave the example of $$ X = \emptyset $$ and $$ R = \emptyset $$)**

- 1 pts Unclear or imprecise mathematical statements made. For example, the argument did not give a clear explanation of both properties, or gave some correct examples, but also some incorrect examples.

- 2 pts Missing or incorrect example, or major misunderstandings

Problem 3. Let n be a positive natural number. Let $X = \{i \in \mathbb{N} : 1 \le i \le n\}$. Denote by $\mathcal{P}(X)$ the power set of X, and let $\mathcal{P}^\star(X) := \mathcal{P}(X) \setminus \{\emptyset\}$ denote the set of subsets of X that are not empty. Consider the function

$$f: \mathcal{P}^\star(X) \to X$$

which sends each non-empty subset of X to its least element. For instance, $f(\{1,3\}) = 1$. For which values of n is f injective, surjective, or bijective? Carefully motivate your arguments.

For f to be Injective, n must be 1. This is because even starting with n=2 (the next case after 1), f is already not Injective and the number of mappings that show f is not injective only increases as n increases.

for $n=2$: $X=\{1,2\}$

$P^*(X) = \{1\}, \{2\}, \{1,2\}$

$\{1\} \longrightarrow 1$ shows f is not injective

$\{1,2\} \longrightarrow 1$

for $n=1$: $X=\{1\}$

$P^*(X) = \{1\}$

$\{1\} \rightarrow 1$ f is injective

$\therefore$, f is only injective when $n=1$

For f to be surjective, n can be any positive natural number. Let's remember that a function f is surjective when cod(f) = ran(f). The reason n can be any positive natural number is because the single element subsets in $P^*(X)$, make cod(f) = ran(f). * continued on back

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Let's take a look at an example. If $n=3$, the set $X=\{1,2,3\}$, which is also the codomain. Within $P^*(X)$, we have ...

$$\underbrace{\{1\},\{2\},\{3\}}, \{1,2\}, \{2,3\}, \{1,3\}, \{1,2,3\}$$

Let's take a look a the mappings of the single element subsets.

$$\{1\} \to 1$$
$$\{2\} \to 2$$
$$\{3\} \to 3$$

As we see, we have mapped to 1, 2, 3 which is actually the entire codomain of f, making f surjective. This logic can actually apply to any positive natural number n, because the single element subsets help "fill up" the codomain. Thus, f is surjective for all positive natural numbers.

Bijective is both injective and surjective. Thus, we need to find the overlap, or intersection, of n for the injective and surjective parts.

Because f is only injective on $n=1$ and f is surjective for all positive natural numbers, <u>f is only bijective when $n=1$.</u>

3 Function Question 3 / 3

✓ **- 0 pts** **Correct**

- **- 1 pts** incomplete or incorrect argument for injectivity when n = 1
- **- 1 pts** incomplete or incorrect argument for surjectivity
- **- 1 pts** incomplete or incorrect argument for non-injectivity for n >1

Problem 4. A teacher wants to arrange their 17 students in a single line. There are two students Averie and Charlie, in this class. How many ways are there for the students to line up so that either Averie is first in line or Charlie is last (or both)?

solve the problem below using patterns on smaller groups of students:

~~pretend only~~

pretend only 3 students: A, B, C
6 permutations

ABC ACB BAC BCA CAB CBA

pretend only 4 students: A, B, C, D
24 permutations

ABCD ACBD ABDC ADBC ADCB ACDB

BACD BADC BDAC BDCA BCAD BCDA

CABD CADB CBDA CBAD CDBA CDAB

DABC DACB DCBA DCAB DBAC DBCA

Identify when we have a desired case:
A= Averie, C= Charlie, B and D don't matter

(oval) = a permutation where Averie is in front
(rectangle) = a permutation where Charlie is in back.

we see that the number of times Avery is in front w/ 3 students is 2/6 → 1/3

" " Avery is in front w/ 4 students is 6/24 → 1/4

* continued in back

*

we see with both the fractions 1/3 and 1/4, that the ratio of the orders where Avery is in front compared to the total number of line orders is just 1/n in each respective case, where n = the number of students in line. The same logic can be applied to Charlie. However, these fractions just give us a ratio, to get the number of occurrences, we multiply by the total # of permutations, or n! Thus, we can write an equation that shows the number of ways for students to line up so that either Averie is first or Charlie is last:

$$n!\left(\frac{1}{n}+\frac{1}{n}\right) \rightarrow n!\left(\frac{2}{n}\right)$$

However, we also notice that there are items we double counted. This occurs when both Averie is in front and Charlie is in back. If we imagine, "locking" Averie and Charlie in place, the number of repeats is just the number of permutations you can make between Averie and Charlie.

$$\underline{A}\ \overbrace{\_\ \_\ \_}\ \underline{C}$$ how many permutations here?

That number is just $(n-2)!$ Thus, we just subtract that expression to get a final equation:

$$n!\left(\frac{2}{n}\right)-(n-2)!$$

for 17 students:

$$17!\left(\frac{2}{17}\right) - 15! = \boxed{40{,}537{,}905{,}408{,}000 \text{ different ways}}$$

$$\downarrow$$

$$\boxed{2(16!) - 15!} =$$

4 Counting Question 2 / 2

✓ **- 0 pts** **Correct**

- **- 0.5 pts** 16! ways with Averie first and 16! ways with Charlie last
- **- 0.5 pts** 15! ways with Averie first and Charlie last
- **- 1 pts** 2(16!)-15! total by Inclusion-Exclusion Principle