61 Midterm 1

Kendrake Jay Tsui

TOTAL POINTS

52 / 70

QUESTION 1

1 Exercise 1 **6 / 10**

- **- 2 pts** (1) Incorrect
- ✓ **- 2 pts (2) Incorrect**
- **- 2 pts** (3) Incorrect
- **- 2 pts** (4) Incorrect
- ✓ **- 2 pts (5) Incorrect**
- **- 0 pts** Fully correct

5) Let X be the empty set.

QUESTION 2

2 Exercise 2 **10 / 10**

- **- 8 pts** Only basis case correct
- **- 8 pts** only demonstrated understanding of Sigma notation
- **- 4 pts** Did not complete inductive step correctly
- **- 2 pts** algebra/logic errors
- **- 1 pts** Minor arithmetic errors
- **- 1 pts** Structure of the proof not made clear (e.g. induction hypothesis not mentioned explicitly, or not clear how induction hypothesis was used).
- ✓ **- 0 pts fully correct**
- **- 10 pts** Not graded

QUESTION 3

3 Exercise 3 **10 / 10**

- **- 10 pts** Incorrect without partial solution
- **- 8 pts** Drew Venn diagram correct but written proof incorrect, or some informal argument
- **- 8 pts** Gives an argument that only works for finite sets
- **- 8 pts** Mixes up union with intersection or Cartesian product
- **- 5 pts** Contains parts of a valid formal approach
- ✓ **- 0 pts Fully correct**
- **- 10 pts** Skipped

QUESTION 4

4 Exercise 4 **10 / 10**

- ✓ **- 0 pts Completely correct**
- **- 3 pts** Incorrect basis case
- **- 4 pts** Correctly implemented induction but didn't succeed in inductive step
- **- 1 pts** Minor arithmetic errors
- **- 6 pts** Correct basis case but did not set up inductive step
- **- 10 pts** skipped

QUESTION 5

5 Exercise 5 **0 / 10**

- ✓ **- 10 pts Not graded**
- **- 5 pts** (1) Incorrect/blank
- **- 4 pts** (1) minimal progress
- **- 4 pts** (1) Diagram or example only
- **- 3 pts** (1) Vague explanation/missing or incorrect steps
- **- 1 pts** (1) Mostly correct, but lacking detail, or disorganized.
- **- 1 pts** (1) Used "X subset Y --> f(X) subset f(Y)" without proof
- **- 5 pts** (2) Incorrect/blank
- **- 4 pts** (2) minimal progress (e.g. just writing the definition of injective.)
- **- 3 pts** (2) Vague explanation/missing steps
- **- 2 pts** (2) Partially correct, but some steps missing
- **- 1 pts** (2) Mostly correct, but unclear or lacking detail.
- **- 0 pts** fully correct

QUESTION 6

6 Exercise 6 **6 / 10**

- **0 pts** correct
- **2 pts** (1) Demonstrated understanding of problem but did not give correct solution
- **1 pts** (1) Failed to show all 3 conditions of an equivalence rel

✓ **- 2 pts (2) Incorrect solution but demonstrated understanding of equivalence class**

- **5 pts** (1) Blank
- **5 pts** (2) Blank
- **10 pts** Not Graded
- **8 pts** Incorrect and did not demonstrate understanding of problem

- 2 Point adjustment

You cannot assume the statements are true and then prove them

QUESTION 7

7 Exercise 7 **10 / 10**

✓ **- 0 pts Correct**

- **2 pts** Arithmetic errors
- **10 pts** Incorrect answer with no work
- **10 pts** Not graded
- **5 pts** Miscalculate tn

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MATH 61 - MIDTERM EXAM 1

0.1. **Instructions.** This is a 50 minute exam. You should feel free to quote any theorems proved in class, as well as anything proved in the homework or discussion section. There are 7 questions—on the real exam, you are required to do the first true/false question, and choose 5 of the remaining 6. Only 5 problems other than the true/false question will be graded so *you should indicate which problems you want graded*, in the case that you attempt all 6. Each question is worth 10 points. Unless otherwise specified, you are required to justify your answers.

Exercise 0.1. Indicate whether the following statements are true or false. You do not need to justify your answer.

(1) If $f : X \to Y$ and $g : Y \to Z$ are one-to-one functions, then $(g \circ f) : X \to Z$ is one-to-one.
(2) If $f : X \to Y$ and $g : Y \to Z$ are onto functions, then $(g \circ f) : X \to Z$ is onto.
(3) If R is a relation on X, then R is symmetric if and only if $R = R^{-1}$.
(4) If R is a reflexive relation on X, then R is transitive if and only if $R \circ R = R$.
(5) If X is a subset of Y then X and Y are not disjoint.

1) True
2) False
3) True
4) True
5) True

Date: October 13, 2018; Ramsey.

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Grade

Exercise 0.2. Show that for all natural numbers n,

$$\sum_{i=1}^{n}(i+1)2^i = n2^{n+1}.$$

WTS: $(n+1)2^{n+2}$

Base: $n=1 \Rightarrow \sum_{1}^{1}(1+1)2^{(1)} = (1)2^{(1)+1}$

$4 = 4$ ✓

Ind. Hyp: Assume $\sum_{i=1}^{k}(i+1)2^i = k2^{k+1}$ for some $k \geq 1$.

Proof: $\sum_{i=1}^{k+1}(i+1)2^i = \sum_{i=1}^{k}(i+1)2^i + (k+2)2^{k+1}$

$= k2^{k+1} + (k+2)2^{k+1}$

$= k2^{k+1} + k2^{k+1} + 2^{k+2}$

$= 2k2^{k+1} + 2^{k+2}$

$= k2^{k+2} + 2^{k+2}$

$= (k+1)2^{k+2}$

So by Mathematical Induct

$\sum_{i=1}^{n}(i+1)2^i = n2^{n+1}$ for $n \geq 1$ □

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Exercise 0.3. Prove If B_1, B_2, C_1, C_2 are sets and $B_1 \subseteq C_1$ and $B_2 \subseteq C_2$ then
$$B_1 \cup B_2 \subseteq C_1 \cup C_2.$$

Since $B_1 \subseteq C_1$, there is some $b_1 \in B_1$ that is also $b_1 \in C_1$.
Since $B_2 \subseteq C_2$, there is some $b_2 \in B_2$ that is also $b_2 \in C_2$.
This holds true for all elements in B_1 and B_2 respectively.
$B_1 \cup B_2$ = all b_0 and $b_1 \in B_1 \cup B_2$ and since b_0 and $b_1 \in C_1$ and C_2 respectively. $b_0, b_1 \in C_1 \cup C_2$. Since b_0, b_1 are also contained in $C_1 \cup C_2$, $B_1 \cup B_2$ is a subset of $C_1 \cup C_2$.
Thus $B_1 \cup B_2 \subseteq C_1 \cup C_2$. $\square$

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Exercise 0.4. Show that $n^2 - 7n + 13$ is nonnegative for all natural numbers $n \geq 3$.

Base: $n=3 \Rightarrow 3^2 - 7(3) + 13 = 9 - 21 + 13 = 1 > 0$ ✓

Induct. Hyp: Assume $k^2 - 7k + 13 > 0$ for some $k \geq 3$.

Proof: $(k+1)^2 - 7(k+1) + 13 = k^2 + 2k + 1 - 7k + 1 + 13 > k^2 - 7n + 13 > 0$ for $k \geq 3$.

By mathematical induction, $n^2 - 7n + 13 > 0$ for $n \geq 3$. $\square$

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Do NOT Grade

Exercise 0.5. Let $f : X \to Y$ be a function. Given any subset $S \subseteq X$, we write $f(S)$ for the set defined as follows: $T \subseteq X$

$$f(S) = \{y \in Y : \text{ there is } s \in S \text{ such that } f(s) = y\}.$$

(1) Show that $f(S \cap T) \subseteq f(S) \cap f(T)$.
(2) Show that if f is injective, then $f(S \cap T) = f(S) \cap f(T)$.

Don't Grade

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Grade

Exercise 0.6. Suppose X and Y are sets.

(1) Suppose E_1 and E_2 are equivalence relations on X and Y respectively. Define a relation E on $X \times Y$ by
$$(x_1, y_1)E(x_2, y_2) \iff x_1 E_1 x_2 \text{ and } y_1 E_2 y_2.$$
Show E is an equivalence relation.

(2) Let S be the set of equivalence classes of E. Show that
$$S = \{[x]_{E_1} \times [y]_{E_2} : x \in X, y \in Y\},$$
where $[x]_{E_1}$ is the equivalence class of x with respect to the equivalence relation E_1 and $[y]_{E_2}$ is the equivalence class of y with respect to the equivalence relation E_2.

1) Reflex: Assume E is reflexive so $(x_1,y_1)E(x_1,y_1)$. Since E_1 and E_2 are equivalence relations, $x_1E_1x_1$ and $y_1E_2y_1$. Therefore $(x_1,y_1)E(x_1,y_1)$ exists and E is reflexive.

Transitive: Assume E is transitive so if $(x_1,y_1)E(x_2,y_2)$ and $(x_2,y_2)E(x_3,y_3)$, then $(x_1,y_1)E(x_3,y_3)$. E_1 is an equivalence relation so if $x_1E_1x_2$ and $x_2E_1x_3$, $x_1E_1x_3$. The same is true with E_2 and y_1, y_2, y_3. So $y_1E_2y_3$ exists. Since $x_1E_1x_3$ and $y_1E_2y_3$, $(x_1,y_1)E(x_3,y_3)$ exists so E is transitive.

Symmetric: Assume E is symmetric so if $(x_1,y_1)E(x_2,y_2)$, $(x_2,y_2)E(x_1,y_1)$.
If $x_1E_1x_2$ then $x_2E_1x_1$ because E_1 is an equivalence relation and if $y_1E_2y_2$, $y_2E_2y_1$ because E_2 is an equivalence relation. Since $x_1E_1x_2$, $x_2E_1x_1$, $y_1E_2y_2$, $y_2E_2y_1$ $(x_1,y_1)E(x_2,y_2)$ and $(x_2,y_2)E(x_1,y_1)$ and E is symmetric.

Thus E is an equivalence relation since it is reflexive, transitive, and symmetric.

2) Since E_1 is an equivalence relation on X_1 any $x_0 \in [x]_{E_1}$. The same is true for E_2 on Y so any $y_0 \in [y]_{E_2}$. Since E is a relation on $X \times Y$, it takes $x_0 \in X$ and $y_0 \in Y$. Since $x_0 \in [x]_{E_1}$ and $y_0 \in [y]_{E_2}$, the set of the equivalence classes of $E = \{[x]_{E_1} \times [y]_{E_2} : x \in X, y \in Y\}$. □

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Exercise 0.7. Define a sequence by $t_1 = 2$ and $t_n = \prod_{i=1}^{n-1} t_i$ for all $i \geq 2$. Define an additional sequence by $s_n = \sum_{i=1}^{n} t_i$. Calculate s_3 and t_4.

$t_1=2$
$t_2=2$
$t_3=4$
$t_4=16$

$\boxed{t_4=16}$

$\boxed{s_3=8}$

$s_1=2$
$s_2=4$
$s_3=8$