61 Midterm 1

Nathan George Midkiff

TOTAL POINTS

60 / 70

QUESTION 1

1 Exercise 1 **10 / 10**

- **- 2 pts** (1) Incorrect
- **- 2 pts** (2) Incorrect
- **- 2 pts** (3) Incorrect
- **- 2 pts** (4) Incorrect
- **- 2 pts** (5) Incorrect
- ✓ **- 0 pts Fully correct**

QUESTION 2

2 Exercise 2 **10 / 10**

- **- 8 pts** Only basis case correct
- **- 8 pts** only demonstrated understanding of Sigma notation
- **- 4 pts** Did not complete inductive step correctly
- **- 2 pts** algebra/logic errors
- **- 1 pts** Minor arithmetic errors
- **- 1 pts** Structure of the proof not made clear (e.g. induction hypothesis not mentioned explicitly, or not clear how induction hypothesis was used).
- ✓ **- 0 pts fully correct**
- **- 10 pts** Not graded

QUESTION 3

3 Exercise 3 **10 / 10**

- **- 10 pts** Incorrect without partial solution
- **- 8 pts** Drew Venn diagram correct but written proof incorrect, or some informal argument
- **- 8 pts** Gives an argument that only works for finite sets
- **- 8 pts** Mixes up union with intersection or Cartesian product
- **- 5 pts** Contains parts of a valid formal approach
- ✓ **- 0 pts Fully correct**
- **- 10 pts** Skipped

QUESTION 4

4 Exercise 4 **10 / 10**

- ✓ **- 0 pts Completely correct**
- **- 3 pts** Incorrect basis case
- **- 4 pts** Correctly implemented induction but didn't succeed in inductive step
- **- 1 pts** Minor arithmetic errors
- **- 6 pts** Correct basis case but did not set up inductive step
- **- 10 pts** skipped

QUESTION 5

5 Exercise 5 **10 / 10**

- **- 10 pts** Not graded
- **- 5 pts** (1) Incorrect/blank
- **- 4 pts** (1) minimal progress
- **- 4 pts** (1) Diagram or example only
- **- 3 pts** (1) Vague explanation/missing or incorrect steps
- **- 1 pts** (1) Mostly correct, but lacking detail, or disorganized.
- **- 1 pts** (1) Used "X subset Y --> f(X) subset f(Y)" without proof
- **- 5 pts** (2) Incorrect/blank
- **- 4 pts** (2) minimal progress (e.g. just writing the definition of injective.)
- **- 3 pts** (2) Vague explanation/missing steps
- **- 2 pts** (2) Partially correct, but some steps missing
- **- 1 pts** (2) Mostly correct, but unclear or lacking detail.
- ✓ **- 0 pts fully correct**

QUESTION 6

6 Exercise 6 **0 / 10**

- **- 0 pts** correct
- **- 2 pts** (1) Demonstrated understanding of problem

but did not give correct solution

- **- 1 pts** (1) Failed to show all 3 conditions of an equivalence rel
- **- 2 pts** (2) Incorrect solution but demonstrated understanding of equivalence class
- **- 5 pts** (1) Blank
- **- 5 pts** (2) Blank
- ✓ **- 10 pts Not Graded**
- **- 8 pts** Incorrect and did not demonstrate understanding of problem

QUESTION 7

7 Exercise 7 **10 / 10**

- ✓ **- 0 pts Correct**
- **- 2 pts** Arithmetic errors
- **- 10 pts** Incorrect answer with no work
- **- 10 pts** Not graded
- **- 5 pts** Miscalculate tn

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won't grade #6

MATH 61 - MIDTERM EXAM 1

0.1. **Instructions.** This is a 50 minute exam. You should feel free to quote any theorems proved in class, as well as anything proved in the homework or discussion section. There are 7 questions—on the real exam, you are required to do the first true/false question, and choose 5 of the remaining 6. Only 5 problems other than the true/false question will be graded so *you should indicate which problems you want graded*, in the case that you attempt all 6. Each question is worth 10 points. Unless otherwise specified, you are required to justify your answers.

Exercise 0.1. Indicate whether the following statements are true or false. You do not need to justify your answer.

(1) If $f : X \to Y$ and $g : Y \to Z$ are one-to-one functions, then $(g \circ f) : X \to Z$ is one-to-one.
(2) If $f : X \to Y$ and $g : Y \to Z$ are onto functions, then $(g \circ f) : X \to Z$ is onto.
(3) If R is a relation on X, then R is symmetric if and only if $R = R^{-1}$.
(4) If R is a reflexive relation on X, then R is transitive if and only if $R \circ R = R$.
(5) If X is a subset of Y then X and Y are not disjoint.

1) True
2) True
3) True
4) True
5) False

Date: October 13, 2018; Ramsey.

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Grade

Exercise 0.2. Show that for all natural numbers n,

$$\sum_{i=1}^{n}(i+1)2^i = n2^{n+1}.$$

Basis: $n=1$

$\sum_{i=1}^{1}(i+1)2^i = (1+1)\cdot 2^1 = 4$

$1\cdot 2^{1+1} = 4$

Induction: Assume it holds true for $n=k$, i.e. $\sum_{i=1}^{k}(i+1)2^i = k2^{k+1}$

$\sum_{i=1}^{k+1}(i+1)2^i = (k+1+1)2^{k+1} + \sum_{i=1}^{k}(i+1)2^i = (k+2)2^{k+1} + k2^{k+1}$

$= (2k+2)2^{k+1} = (k+1)\cdot 2\cdot 2^{k+1} = (k+1)2^{(k+1)+1}$

Because the statement holds for $n=1$, and holds for $k+1$ if it holds for k, then it holds for all natural numbers n.

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Exercise 0.3. If B_1, B_2, C_1, C_2 are sets and $B_1 \subseteq C_1$ and $B_2 \subseteq C_2$ then
$$B_1 \cup B_2 \subseteq C_1 \cup C_2.$$

Let $a \in B_1 \cup B_2$ be arbitrary.
Then $a \in B_1$ or $a \in B_2$, by definition. Because $B_1 \subseteq C_1$ and $B_2 \subseteq C_2$, all the elements in B_1 are in C_1 and all the elements in B_2 are in C_2, so $a \in C_1$ or $a \in C_2$.
So $a \in C_1 \cup C_2$. Because a was an arbitrary member of $B_1 \cup B_2$,
$B_1 \cup B_2 \subseteq C_1 \cup C_2$

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Exercise 0.4. Show that $n^2 - 7n + 13$ is nonnegative for all natural numbers $n \geq 3$.

Basis: $n = 3$

$3^2 - 7 \cdot 3 + 13 = 9 - 21 + 13 = 0 \geq 0$

Induction: Assume $k^2 - 7k + 13 \geq 0$

$(k+1)^2 - 7(k+1) + 13 = k^2 + 2k + 1 - 7k - 7 + 13 = (k^2 - 7k + 13) + 2k - 6 \geq 2k - 6 \geq 0$ for all $n \geq 3$.

$\Rightarrow (k+1)^2 - 7(k+1) + 13 \geq 0$

Because $n^2 - 7n + 13$ is nonnegative for $n = 3$, and for $k+1$ if it is for k, then it is true for all natural numbers $n \geq 3$

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Exercise 0.5. Let $f : X \to Y$ be a function. Given any subset $S \subseteq X$, we write $f(S)$ for the set defined as follows:

$$f(S) = \{y \in Y : \text{ there is } s \in S \text{ such that } f(s) = y\}.$$

(1) Show that $f(S \cap T) \subseteq f(S) \cap f(T)$.
(2) Show that if f is injective, then $f(S \cap T) = f(S) \cap f(T)$.

1) Let $a \in f(S \cap T)$ be arbitrary. By definition, there is some $s \in S \cap T$ s.t. $f(s) = a$. Because $s \in S \cap T$, $s \in S$ and $s \in T$. There fore there is $s \in S$ s.t. $f(s) = a$, so $a \in f(S)$, and there is $s \in T$ s.t. $f(s) = a$, so $a \in f(T)$. Because $a \in f(S)$ and $a \in f(T)$, $a \in f(S) \cap f(T)$. Because a was an arbitrary member of $f(S \cap T)$, $f(S \cap T) \subseteq f(S) \cap f(T)$

2) Let $a \in f(S) \cap f(T)$ be arbitrary, so $a \in f(S)$ and $a \in f(T)$. By definition, there is $s \in S$ s.t. $f(s) = a$, and $t \in T$ s.t. $f(t) = a$. Because f is injective, since $f(s) = f(t)$, $s = t$. There fore $s = t \in S$ and $s = t \in T$, so $s = t \in S \cap T$. Therefore there is $s \in S \cap T$ s.t. $f(s) = a$. So $a \in f(S \cap T)$. Because a was an arbitrary member of $f(S) \cap f(T)$, $f(S) \cap f(T) \subseteq f(S \cap T)$. Because, from (1), $f(S \cap T) \subseteq f(S) \cap f(T)$, $f(S \cap T) = f(S) \cap f(T)$

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Do Not Grade

Exercise 0.6. Suppose X and Y are sets.

(1) Suppose E_1 and E_2 are equivalence relations on X and Y respectively. Define a relation E on $X \times Y$ by
$$(x_1, y_1)E(x_2, y_2) \iff x_1 E_1 x_2 \text{ and } y_1 E_2 y_2.$$
Show E is an equivalence relation.

(2) Let $\mathcal{S}$ be the set of equivalence classes of E. Show that
$$\mathcal{S} = \{[x]_{E_1} \times [y]_{E_2} : x \in X, y \in Y\},$$
where $[x]_{E_1}$ is the equivalence class of x with respect to the equivalence relation E_1 and $[y]_{E_2}$ is the equivalence class of y with respect to the equivalence relation E_2.

1) symmetry: if $(x_1, y_1) E (x_2, y_2)$ then $x_1 E_1 x_2$ and $y_1 E_2 y_2$. Because E_1, E_2 are equiv. relations, $x_2 E_1 x_1$ and $y_2 E_2 y_1$, so $(x_2, y_2) E (x_1, y_1)$

reflexive: $x_1 E_1 x_1$ and $y_1 E_2 y_1$, so $(x_1, y_1) E (x_1, y_1)$

transitive: If $(x_1, y_1) E (x_2, y_2)$ and $(x_2, y_2) E (x_3, y_3)$, then $x_1 E_1 x_2$ and $y_1 E_2 y_2$ and $x_2 E_1 x_3$ and $y_2 E_2 y_3$, so because E_1, E_2 are equiv relations, $x_1 E_1 x_3$ and $y_1 E_2 y_3$, so $(x_1, y_1) E (x_3, y_3)$

2) Let $a \in S$ be arbitrary.

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Exercise 0.7. Define a sequence by $t_1 = 2$ and $t_n = \prod_{i=1}^{n-1} t_i$ for all $i \geq 2$. Define an additional sequence by $s_n = \sum_{i=1}^{n} t_i$. Calculate s_3 and t_4.

$$S_3 = \sum_{i=1}^{3} t_i = t_1 + t_2 + t_3 = t_1 + \prod_{n=1}^{1} t_i + \prod_{n=1}^{2} t_i = t_1 + t_1 + t_1 \cdot t_2 = t_1 + t_1\ t\ t_1 \cdot t_1$$

$$= 2 + 2 + 2 \cdot 2 = \boxed{8}$$

$$t_2 = \prod_{n=1}^{1} t_i = t_1 = 2$$

$$t_3 = \prod_{n=1}^{2} t_i = t_1 \cdot t_2 = 2 \cdot 2 = 4$$

$$t_4 = \prod_{n=1}^{3} t_i = t_1 \cdot t_2 \cdot t_3 = 2 \cdot 2 \cdot 4 = 16$$

$$\boxed{t_4 = 16}$$