**Instructions:**

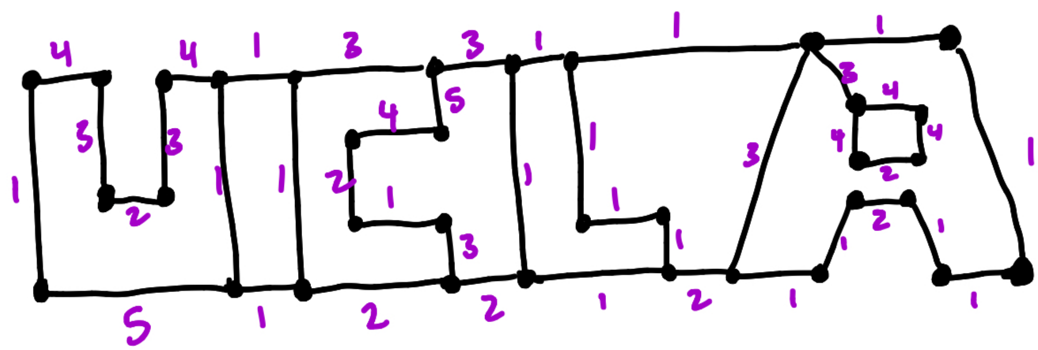
- You have from Wednesday June 9 at 8:00am to Thursday June 10 8:am Pacific Time to solve this exam.
- Scan or type your solutions and upload them to Gradescope. You should submit readable scans, and not pictures of your solutions (you can use a scanner app on your phone, for instance). Please make sure to match the problems on the exam template with the respective parts in your solutions.
- This exam is open book, and you are allowed to use the textbook, and all resources from the lecture, or similar resources.
- You are not allowed to ask for help from others, nor give help to others taking this exam. Students suspected of academic dishonesty may be reported to the Dean of Students.
- Show your work. Full points are only given for correct answers *with* adequate justification. A correct final answer with missing or substantially incorrect justification will not merit full points on a problem.
- On any question asking you to calculate a number, you may leave your final answer either in combinatorial notation (using factorials and combinations, etc.) or a precise numerical value.
- All graphs are assumed to be simple.

**Code of honour**

Academic integrity is of the uttermost importance. By taking part in this evaluation, you are accepting the following code of honor:

*I certify on my honor that I have neither given nor received any help, or used any non-permitted resources, while completing this evaluation.*

**Problem 1.** Consider the following graph:



What is the weight of a minimal spanning tree for this graph?

**Problem 2.**

1. Suppose $H$ is an acyclic graph (i.e. a graph with no cycles). Show that every connected component of $H$ is a tree.

2. Suppose $0 < k < n$, where $n$ and $k$ are natural numbers. Show that a graph $G$ with $n$ vertices and $n - k$ edges must have at least $k$ distinct connected components.

**Problem 3.** Let $V_1 = \{1,2,3,4,5\}$ and $V_2 = \mathcal{P}(\{1,2,3,4,5\})$, that is, the power set of $\{1,2,3,4,5\}$. Define a bipartite graph $G$ whose vertices are $V = V_1 \cup V_2$ such that, for each $i \in \{1,2,3,4,5\}$ and $X \in \mathcal{P}(\{1,2,3,4,5\})$, there is an edge between $i$ and $X$ if and only if $i \in X$ (and no edge between any two elements of $V_1$ and no edge between any two elements of $V_2$).

1. What is the degree of each vertex $v \in V_1$?

2. How many edges are there in $G$?

3. Show that $G$ is not planar. (*Hint*: Show that $K_{3,3}$ is a subgraph of $G$)

**Problem 4.** Consider the recurrence relation

$$3a_n - 12a_{n-2} = 0$$

for all $n \geq 2$. Find the solution to this recurrence relation that satisfies the initial conditions $a_0 = a_1 = 2$.

**Problem 5.** For each non-negative integer $n$, let $T_n$ be a rooted tree of height $n$ with the property that every vertex at a level $< n$ has exactly 4 children.

1. Let $s_n$ be the number of terminal vertices of $T_n$. Give an expression for $s_n$ in terms of $n$ for all non-negative integers $n$.

2. Let $t_n$ be the total number of vertices in $T_n$. Write a recurrence relation for $t_n$.

3. Give an expression for $t_n$ in terms of $n$ for all non-negative integers $n$.

**Problem 6.** Recall that $K_n$ denotes the complete graph on $n$ vertices.

1. Show that, if weights from the set $\{1, \ldots, n\}$ are assigned to the edges of $K_n$ and $n \geq 4$, then there must be at least two edges with the same weight.

2. What is the least $n$ such that, no matter how weights are assigned to the edges of $K_n$ from the set $\{1, \ldots, n\}$, there must be at least 11 edges with the same weight?

**Problem 7.**

1. How many ways are there to order the letters of the word

   CALIFORNIA

   to get a distinct word?

2. How many ways are there to order the letters of the word

   CALIFORNIA

   to get a distinct word, subject to the requirement that the A's are not adjacent *or* the I's are not adjacent (the *or* here is inclusive)?

**Problem 8.** Suppose $A$ is a set and $\sigma : A \to A$ is a bijection. Given any $X \subseteq A$, define a function $\tilde{\sigma} : \mathcal{P}(A) \to \mathcal{P}(A)$ by $\tilde{\sigma}(X) = \{\sigma(i) : i \in X\}$.

1. Show that $\tilde{\sigma}$ is a bijection from $\mathcal{P}(A)$ to $\mathcal{P}(A)$.

2. Consider the special case now when $A = \{1, 2, 3, 4, 5\}$ and $\sigma : A \to A$ is defined by $\sigma(i) = i + 1$ for $i = 1, 2, 3, 4$ and $\sigma(5) = 1$. Calculate $\tilde{\sigma}(\{1, 3, 5\})$.

3. Let $G$ be the graph from Problem 3. Let $f : V \to V$ by defined by

$$f(v) = \begin{cases} \sigma(v) & v \in V_1 \\ \tilde{\sigma}(v) & v \in V_2 \end{cases}$$

   where $\sigma : \{1, 2, 3, 4, 5\} \to \{1, 2, 3, 4, 5\}$ is defined as in (2). Show that $f$ is an isomorphism from the graph $G$ to the graph $G$.