Math 61 Final Exam

TOTAL POINTS

40 / 40

QUESTION 1

1 Counting functions 5 / 5

✓ **+ 2 pts** a: $$ 2^8 = 256 $$ options total

✓ **+ 3 pts** b: $$ 2 $$ functions total, by casing on which element of $$ Y $$ is not in the image

+ **0 pts** a: Incorrect count

+ **2 pts** b: Correct count, but did not properly justify some steps

+ **1.5 pts** b: Incorrect count, but some promising progress.

+ **0 pts** b: Incorrect count, as well as several other incorrect statements along the way

QUESTION 2

2 f squared equals the identity 5 / 5

✓ **+ 2 pts** Injectivity: Let $$ x, y \in X $$ be such that $$ f(x) = f(y) $$. Then $$ f(f(x)) = f(f(y)) $$. Thus $$ x = f(f(x)) = f(f(y)) = y $$, so that $$ x = y $$. Hence, for any $$ x, y \in X $$, if $$ f(x) = f(y) $$, we have $$ x = y $$. We conclude $$ f $$ is injective, as desired.

+ **1 pts** Injectivity: Partial progress, but informal or missing some important details

+ **0 pts** Injectivity: No serious progress. Arguments that may fall into this category include: trying to use $$ f^{-1} $$ without justifying $$ f $$ is invertible, trying to claim $$ f $$ is the identity function, trying to use the cardinality of $$ X $$ (e.g. assuming $$ |X|< \infty $$), showing injectivity of $$ f \circ f $$ instead of $$ f $$

✓ **+ 3 pts** Surjectivity: Let $$ y \in X $$ be arbitrary. Notice $$ f(f(y)) = y $$. Thus, there exists an $$ x \in X $$, namely $$ x = f(y) $$, such that $$ f(x) = y $$. Since $$ y \in X $$ was arbitrary, we see that for all $$ y \in X $$, there exists an $$ x \in X $$ such that $$ f(x) = y $$. Thus, $$ f $$ is surjective, as desired.

+ **1.5 pts** Surjectivity: Incorrect or missing quantifiers, but argument is correct or almost correct. An informal but (almost) correct argument may also fall into this category.

+ **0 pts** Surjectivity: No serious progress. Arguments that may fall into this category include: trying to use $$ f^{-1} $$ without justifying $$ f $$ is invertible, trying to claim $$ f $$ is the identity function, trying to use the cardinality of $$ X $$ (e.g. assuming $$ |X|< \infty $$), showing surjectivity of $$ f \circ f $$ instead of $$ f $$

QUESTION 3

3 Equipartitions 5 / 5

✓ **+ 5 pts** Correct

+ **1 pts** Two examples of equipartitions for arbitrary sets with at least two elements.

+ **1 pts** Clear explanations in the first part

+ **0.75 pts** Explanation of examples is almost clear but definitions are not clearly stated.

+ **0.5 pts** Explanation of examples present but is not thorough or otherwise incorrect

+ **1 pts** Correctly identifying partition sizes for equipartitions of sets of nine elements

+ **1 pts** Correct counting of equipartitions

+ **1 pts** Clear explanation of the second part

+ **0.75 pts** Explanation of the second part is understandable but skips essential details or does not use enough words to explain the ideas. Well articulated but incorrect explanations can be awarded this score.

+ **0.5 pts** Explanation of the second part is included but is not thorough or is otherwise incorrect

+ **0.5 pts** Incorrect counting argument but with significant correct steps.

+ **0 pts** No content

QUESTION 4

4 ALIVE **5 / 5**

✓ **- 0 pts** **Correct**

- 2 pts 5! total arrangements (by multiplication principle)

- 1 pts 2 "EVIL" arrangements (AEVIL and EVILA)

- 2 pts 5! - 2 = 118 arrangements (inclusion/exclusion)

QUESTION 5

5 Fully splitting tree **5 / 5**

✓ **+ 5 pts** **Correct**

+ 1 pts Base case or argument accounts for the tree with one vertex

+ 1 pts Decomposing fully splitting trees in the induction step

+ 0.5 pts Builds a fully splitting binary tree in the induction step (but does not address whether or why they are all built this way)

+ 1 pts Correct use of induction hypothesis about fully splitting binary trees

+ 1 pts Counting arguments have accurate numbers and manipulation of equations and/or inequalities

+ 1 pts Clarity

QUESTION 6

6 Solution counting **5 / 5**

✓ **- 0 pts** **Correct**

- 1 pts Use generalized combination (stars and bars)

- 1 pts Start with 2 "stars" in $$x_1$$

- 1 pts Use 3 bars and the remaining 15 stars to get C(18,3) = 816

- 2 pts Remove the 2 bad solutions (2,15,0,0) and (2,0,15,0) to get 814

- 0 pts Click here to replace this description.

QUESTION 7

7 Minimal spanning tree **5 / 5**

✓ **- 0 pts** **Correct**

- 1.5 pts Incorrect degree for (0,0)

- 1.5 pts Incorrect degree for (2,2)

- 2 pts Incorrect weight

QUESTION 8

8 Bipartite **5 / 5**

✓ **- 0 pts** **Correct**

- 2 pts Assumes the graph is *complete* bipartite without justification

- 2 pts Assumes the bipartition has two sets with specific sizes

- 0 pts Click here to replace this description.

Problem 1. Suppose X and Y are sets with $|X| = 8$ and $|Y| = 2$.

1. How many functions are there from X to Y?
2. How many functions are there from X to Y that are neither injective nor surjective? That is, how many functions are there from X to Y that have the property of being not injective and the property of being not surjective?

1. There are **2^8 functions**, since each element in X has 2 choices in Y that it can be mapped to.

2. # of functions that are injective: 0
since each element in X must correspond to one of 2 possible values in Y. We can consider each $x \in X$ as a pigeon and each $y \in Y$ as a pigeonhole. By the Pigeonhole Principle, $\lceil \frac{8}{2} \rceil = 4$, so there are at least 4 $x \in X$ that map to the same y value. Thus, for $f(x) = f(x') = y$, where $x, x' \in X$ and $y \in Y$, it is not guaranteed that $x = x'$ because there are at least 4 values of x that satisfy this property.

Because none of the possible functions are injective, we only need to count the number of functions that are not surjective. In this situation, a function that is not surjective has at least one element $y \in Y$ that is not mapped to. In other words, since there are only 2 choices in Y that an element $x \in X$ can be mapped, a function that is not surjective has all elements $x \in X$ map to the same $y \in Y$ such that there is not element $x \in X$ such that $f(x) = y'$, the other element in Y.
By this reasoning, there are 2 functions that are not surjective, where for every $x \in X$, $y_1, y_2 \in Y$, $f(x) = y_1$ for the 1st function or $f(x) = y_2$ for the second function.

Since all functions from X to Y are not injective and there are 2 functions that are not surjective, there are **2 functions** from X to Y that are neither injective nor surjective.

1 Counting functions **5 / 5**

- ✓ **+ 2 pts** **a: $$ 2^8 = 256 $$ options total**
- ✓ **+ 3 pts** **b: $$ 2 $$ functions total, by casing on which element of $$ Y $$ is not in the image**
- **+ 0 pts** a: Incorrect count
- **+ 2 pts** b: Correct count, but did not properly justify some steps
- **+ 1.5 pts** b: Incorrect count, but some promising progress.
- **+ 0 pts** b: Incorrect count, as well as several other incorrect statements along the way

Problem 2. Suppose $f : X \to X$ is a function that satisfies

$$f(f(x)) = x$$

for all $x \in X$. Show that f is a bijection.

To show that f is a bijection, we want to show that f is injective and surjective

1. Injective

For $x_1, x_2 \in X$, let $f(x_1) = y$ and $f(x_2) = y$.

Then, we have $f(f(x_1)) = x_1$ and $f(f(x_2)) = x_2$

$\Rightarrow f(y) = x_1$ and $f(y) = x_2$

$\Rightarrow x_1 = f(y) = x_2$

$x_1 = x_2$

It has been shown that, if $f(x_1) = f(x_2)$, then $x_1 = x_2 \Rightarrow f$ is injective

2. surjective

For an arbitrary $x \in X$, let $f(x) = y$

Then, we have $f(f(x)) = x$

$\Rightarrow f(y) = x$

f is a function from $X \to X$, so $y \in X$.

Since x was arbitrary, we have shown that for all $x \in X$, there is some $f(x) = y \in X$ such that $f(y) = x$. Thus, every element in X can be mapped to under f, so f is indeed surjective.

$\therefore$, by showing that f is injective and surjective, we have shown that f is a bijection.

2 f squared equals the identity 5 / 5

✓ **+ 2 pts Injectivity: Let $$ x, y \in X $$ be such that $$ f(x) = f(y) $$. Then $$ f(f(x)) = f(f(y)) $$. Thus $$ x = f(f(x)) = f(f(y)) = y $$, so that $$ x = y $$. Hence, for any $$ x, y \in X $$, if $$ f(x) = f(y) $$, we have $$ x = y $$. We conclude $$ f $$ is injective, as desired.**

+ 1 pts Injectivity: Partial progress, but informal or missing some important details

+ 0 pts Injectivity: No serious progress. Arguments that may fall into this category include: trying to use $$ f^{-1} $$ without justifying $$ f $$ is invertible, trying to claim $$ f $$ is the identity function, trying to use the cardinality of $$ X $$ (e.g. assuming $$ |X|< \infty $$), showing injectivity of $$ f \circ f $$ instead of $$ f $$

✓ **+ 3 pts Surjectivity: Let $$ y \in X $$ be arbitrary. Notice $$ f(f(y)) = y $$. Thus, there exists an $$ x \in X $$, namely $$ x = f(y) $$, such that $$ f(x) = y $$. Since $$ y \in X $$ was arbitrary, we see that for all $$ y \in X $$, there exists an $$ x \in X $$ such that $$ f(x) = y $$. Thus, $$ f $$ is surjective, as desired.**

+ 1.5 pts Surjectivity: Incorrect or missing quantifiers, but argument is correct or almost correct. An informal but (almost) correct argument may also fall into this category.

+ 0 pts Surjectivity: No serious progress. Arguments that may fall into this category include: trying to use $$ f^{-1} $$ without justifying $$ f $$ is invertible, trying to claim $$ f $$ is the identity function, trying to use the cardinality of $$ X $$ (e.g. assuming $$ |X|< \infty $$), showing surjectivity of $$ f \circ f $$ instead of $$ f $$

Problem 3. If X is a set, an *equipartition* of X is an equivalence relation on X such that every equivalence class has the same size. In other words, E is an equivalence relation on X if $|[x]_E| = |[y]_E|$ for all $x, y \in X$, where $[x]_E = \{z \in X : (x, z) \in E\}$.

1. Show that if $|X| \geq 2$, then there are at least two different equipartitions of X.
2. Suppose $|X| = 9$. How many equipartitions of X are there? *Hint*: First, figure out the possibilities for the cardinality of the equivalence classes. Then count the number of ways of selecting the elements of the classes.

1. Let us consider the possible partitions of X if $|X| \geq 2$. Let $\{x_1, \ldots, x_n\}$ be the elements of X such that $n = |X|$ and $n \geq 2$. For any value of n, the following partitions are always possible: $\{\{x_1, \ldots, x_n\}\}$ and $\{\{x_1\}, \ldots, \{x_n\}\}$.
Elements in the same subset of partition are related and are $\in E$, the equivalence relation. We can observed that the 1st partition yields $|[x_1]_E| = \ldots = |[x_n]_E| = n$ because all elements of X are in the same subset, thus forming an possible equipartition. The second partition yields $|\{x_n\}| = 1$ for all n, which forms the 2nd equipartition.
$\therefore$, there are at least 2 different equipartitions of X if $|X| \geq 2$.

2. Possible cardinalities $|[x_n]_E| = 1, 3, 9$
$|[x_n]_E| = 1 \Rightarrow$ 1 equipartition (one way of putting 9 elements in 9 groups of 1)
$|[x_n]_E| = 9 \Rightarrow$ 1 equipartition (one way of putting 9 elements in 1 group of 9)

$|[x_n]_E| = 3$
· choose 3 elements to be in 1st equivalence class, 3 in 2nd equivalent class, & 3 in 3rd equivalence class
$$\binom{9}{3}\binom{6}{3}\binom{3}{3} = \binom{9}{3}\binom{6}{3}(1) = \frac{9!}{3!\,6!} \cdot \frac{6!}{3!\,3!} = \frac{9!}{(3!)^3}$$
Since order of these groups do not matter, divide by 3! for the number of permutations of the group
# of equipartitions for $|[x_n]_E| = 3 \Rightarrow \frac{9!}{(3!)^4}$

# of equipartitions of $X = \boxed{\frac{9!}{(3!)^4} + 2}$

3 Equipartitions 5 / 5

✓ + **5 pts** **Correct**

+ **1 pts** Two examples of equipartitions for arbitrary sets with at least two elements.
+ **1 pts** Clear explanations in the first part
+ **0.75 pts** Explanation of examples is almost clear but definitions are not clearly stated.
+ **0.5 pts** Explanation of examples present but is not thorough or otherwise incorrect
+ **1 pts** Correctly identifying partition sizes for equipartitions of sets of nine elements
+ **1 pts** Correct counting of equipartitions
+ **1 pts** Clear explanation of the second part
+ **0.75 pts** Explanation of the second part is understandable but skips essential details or does not use enough words to explain the ideas. Well articulated but incorrect explanations can be awarded this score.
+ **0.5 pts** Explanation of the second part is included but is not thorough or is otherwise incorrect
+ **0.5 pts** Incorrect counting argument but with significant correct steps.
+ **0 pts** No content

Problem 4. How many ways are there to rearrange the letters ALIVE in such a way that the word EVIL is not contained in the resulting word (as a *consecutive* string of letters)?

# of ways to rearrange ALIVE: 5!

# of ways of rearrange ALIVE w/ EVIL:

consider "EVIL" as one letter, so # of ways to arrange "A" & "EVIL" = 2!

# of ways to rearrange ALIVE w/o EVIL = 5! - 2!

= 5! - 2 ways

4 ALIVE 5 / 5

✓ **- 0 pts** **Correct**

- 2 pts 5! total arrangements (by multiplication principle)

- 1 pts 2 "EVIL" arrangements (AEVIL and EVILA)

- 2 pts 5! - 2 = 118 arrangements (inclusion/exclusion)

From the root

Problem 5. Say that a full binary tree is a *fully splitting binary tree* if every non-terminal node has exactly 2 successors and all terminal nodes are at the same level—that is, the length of the unique simple path to a terminal vertex is always the same. Prove that, for all non-negative integers n, if T is a fully splitting binary tree of height n, then T has $2^{n+1} - 1$ vertices.

Proof by induction:

Base case: $n=0$

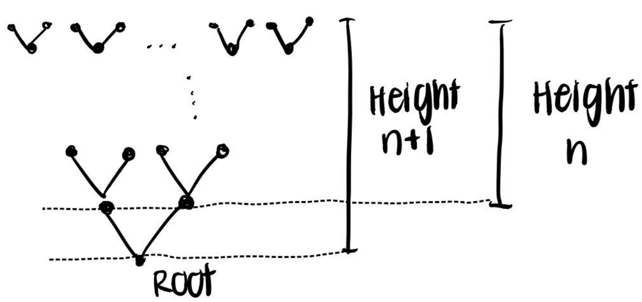
A binary tree with a height of 0 consists of just one vertex, the root

$2^{0+1} - 1 \stackrel{?}{=} 1$

$2^1 - 1 \stackrel{?}{=} 1$

$1 = 1$ ✓

Assume for some height n, where n is a non-negative integer, that a fully splitting binary tree T has $2^{n+1} - 1$ vertices. We want to show that a fully splitting binary tree of height $n+1$ has $2^{(n+1)+1} - 1$ vertices.



If v is a child of the root, then Tv, the subtree of T rooted at v, is a rooted fully splitting binary tree of height n. Since T is a fully splitting binary tree, the subtree Tv is also a fully splitting binary tree b/c every non-terminal node is guaranteed to have exactly 2 successors and have all te[rmi]nal nodes at the same level.

Thus, the two children of the root form 2 fully splitting binary trees of height n. By the induction hypothesis, these subtrees each have $2^{n+1} - 1$ vertices. We can add the number of vertices in these subtrees and the root vertex to get the total # of vertices in the tree.

$\text{vertices} = 2 \cdot (2^{n+1} - 1) + 1$

$= 2^1 \cdot 2^{n+1} - 2 + 1$

$= 2^{(n+1)+1} - 1$ ✓

We have shown by induction that a fully splitting binary tree of height n has $2^{n+1} - 1$ total vertices.

5 Fully splitting tree 5 / 5

- ✓ **+ 5 pts** **Correct**
- **+ 1 pts** Base case or argument accounts for the tree with one vertex
- **+ 1 pts** Decomposing fully splitting trees in the induction step
- **+ 0.5 pts** Builds a fully splitting binary tree in the induction step (but does not address whether or why they are all built this way)
- **+ 1 pts** Correct use of induction hypothesis about fully splitting binary trees
- **+ 1 pts** Counting arguments have accurate numbers and manipulation of equations and/or inequalities
- **+ 1 pts** Clarity

Problem 6. How many solutions are there to the equation

$$x_1 + x_2 + x_3 + x_4 = 17$$

where x_1, x_2, x_3, and x_4 are non-negative integers satisfying $x_1 \geq 2$, $x_2 \leq 14$, and $x_3 \leq 14$.

let $x_1' = x_1 - 2$ for the constraint $x_1 \geq 2$

$x_1' + x_2 + x_3 + x_4 = 17 - 2$

$x_1' + x_2 + x_3 + x_4 = 15$

$C(15+4-1, 4-1) = C(18, 3)$

$= \binom{18}{3}$

Now, find number of bad solutions for x_2 & x_3, or those that fail the constraints $x_2 \leq 14$ and $x_3 \leq 14$.

We want to find the solutions to $x_1' + x_2 + x_3 + x_4 = 15$ such that $x_2 \geq 15$ or $x_3 \geq 15$

Note that there is only 1 solution that satisfies $x_2 \geq 15$. If $x_2 = 15$, x_1', x_3, and x_4 must equal 0 for the sum to be 15. x_2 cannot be greater than 15 while $x_1 \geq 2$ because that would yield a sum > 17. Similarly, there is one solution where $x_3 = 15$ by the same reasoning.

So, there are 2 bad solutions to the total solutions for $x_1' + x_2 + x_3 + x_4 = 15$

Thus, the number of solutions = total − bad solutions

$= \binom{18}{3} - 2$ solutions

6 Solution counting 5 / 5

✓ **- 0 pts** **Correct**

- 1 pts Use generalized combination (stars and bars)

- 1 pts Start with 2 "stars" in $$x_1$$

- 1 pts Use 3 bars and the remaining 15 stars to get C(18,3) = 816

- 2 pts Remove the 2 bad solutions (2,15,0,0) and (2,0,15,0) to get 814

- 0 pts Click here to replace this description.

Problem 7. Let G be a graph whose vertices consist of pairs (a, b) where a and b are non-negative integers less than or equal to 3. Say that there is an edge between (a, b) and (c, d) if either $|a - c| = 1$ and $b = d$ or $|b - d| = 1$ and $a = c$, where $|x|$ denotes the absolute value of x. In other words, two vertices are connected by an edge if they differ by exactly 1 in one coordinate and agree on the other.

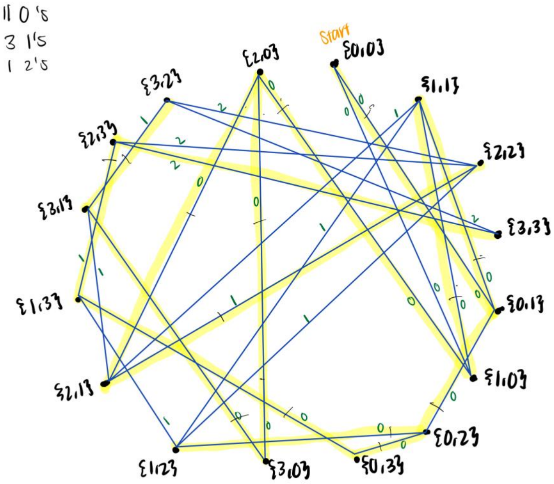
1. What is the degree of $(0, 0)$?
2. What is the degree of $(2, 2)$?
3. Define the weight of an edge between (a, b) and (c, d) to be the minimum of the numbers a, b, c, d. What is the weight of a minimal spanning tree for the graph G?

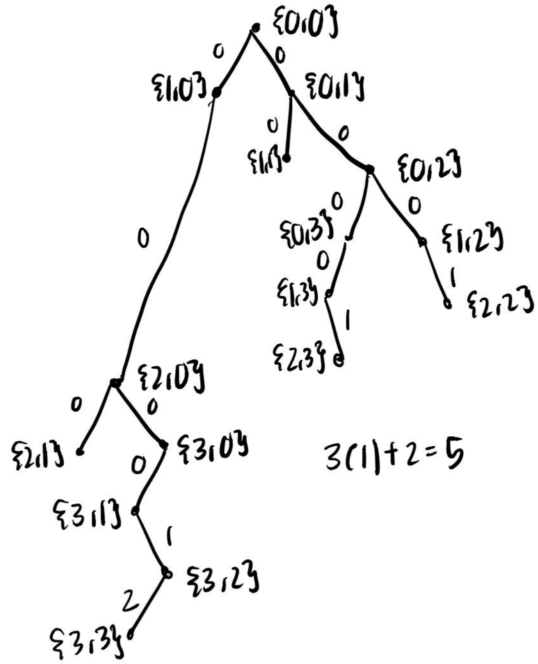
1. deg((0,0)) = 2
   There are only 2 pairs that differ by exactly 1 in one coordinate & agree in the other:
   (0,1) and (1,0)

2. (2,2)
   Differ in 1st coordinate: (1,2), (3,2)
   Differ in 2nd coordinate: (2,1), (2,3)
   deg((2,2)) = 4

3. Weight of minimal spanning tree = 5

11 0's
3 1's
1 2's





3(1)+2=5

7 Minimal spanning tree **5 / 5**

- ✓ **- 0 pts** **Correct**
- **- 1.5 pts** Incorrect degree for (0,0)
- **- 1.5 pts** Incorrect degree for (2,2)
- **- 2 pts** Incorrect weight

Problem 8. Suppose G is a graph with 17 vertices. Show that if G has at least 9 vertices of degree 9, then G cannot be bipartite.

Let G be a graph with at least 9 vertices of degree 9. suppose vertex v_1 has a degree of 9. If G is a bipartite graph $K_{m,n}$, then v_1 is in one partition and there are 9 vertices in partition 2 because there is an edge connecting a vertex in partition 1 to every vertex in partition 2 by definition of a bipartite graph.

From here, there are two scenarios for the rest of the degree-9 vertices:
(1) At least 8 vertices are in the same partition as v_1
(2) At least 1 vertex is in partition 2 (not the same partition as v_1)

In case 1, the size of partition 1 is at least 9, since there are at least 9 vertices (the added 8 + v_1) in partition 1. However, we have already established that there are also 9 vertices in partition 2. Assign the size of partition one to m and the size of partition 2 to n to get $m \geq 9$ and $n = 9$.

$m + n \geq 18 > 17$ vertices.

Thus, this contradicts the number of total vertices in G, so G cannot be bipartite under case 1.

In case 2, we assume that at least 1 vertex is in partition 2. Let this vertex be v_2. Since we know that the $deg(v_2)$ is also 9, there must be 9 vertices in partition 1 such that there is an edge from v_2 to each of the vertices in partition 1. Let m represent the cardinality of partition 1 and n represent the cardinality of partition 2. This gives $m = 9$ and $n = 9$.

$m + n = 18 > 17$ vertices

The number of vertices in such bipartite graph is greater than 17, the number of vertices in G, so G cannot be bipartite under case 2.

By considering these two cases, we have shown that a graph G with 17 vertices and at least 9 vertices with degree 9 cannot be bipartite.

8 Bipartite 5 / 5

✓ **- 0 pts** **Correct**

- 2 pts Assumes the graph is *complete* bipartite without justification

- 2 pts Assumes the bipartition has two sets with specific sizes

- 0 pts Click here to replace this description.