Instructions:

- You have from Tuesday December 15 at 8:00am to Wednesday December 16 8:00am Pacific Time to solve this exam.
- Scan or type your solutions and upload them to Gradescope by Friday 23 October at 11.59pm Pacific Time. You should submit readable scans, and not pictures of your solutions. Please make sure to match the problems on the exam template with the respective parts in your solutions.
- This exam is open book, and you are allowed to use the textbook, and all resources from the lecture, or similar resources.
- You are not allowed to ask for help from others, nor give help to others taking this exam. Students suspected of academic dishonesty may be reported to the Dean of Students.
- Show your work. Full points are only given for correct answers *with* adequate justification. A correct final answer with missing or substantially incorrect justification will not merit full points on a problem.
- On any question asking you to calculate a number, you may leave your final answer either in combinatorial notation (using factorials and combinations, etc.) or a precise numerical value.
- We fix the following mathematical conventions: by *graph*, we mean simple graph (i.e. a graph with no loops and no parallel edges). A path may have length 0 but a cycle must have positive length.

Code of honour

Academic integrity is of the uttermost importance. By taking part in this evaluation, you are accepting the following code of honor:

I certify on my honor that I have neither given nor received any help, or used any non-permitted resources, while completing this evaluation.

Problem 1. Suppose X and Y are sets with $|X| = 8$ and $|Y| = 2$.

1. How many functions are there from X to Y?

2. How many functions are there from X to Y that are neither injective nor surjective? That is, how many functions are there from X to Y that have the property of being not injective and the property of being not surjective?

Problem 2. Suppose $f : X \to X$ is a function that satisfies

$$f(f(x)) = x$$

for all $x \in X$. Show that f is a bijection.

Problem 3. If X is a set, an *equipartition* of X is an equivalence relation on X such that every equivalence class has the same size. In other words, E is an equivalence relation on X if $|[x]_E| = |[y]_E|$ for all $x, y \in X$, where $[x]_E = \{z \in X : (x, z) \in E\}$.

1. Show that if $|X| \geq 2$, then there are at least two different equipartitions of X.

2. Suppose $|X| = 9$. How many equipartitions of X are there? *Hint*: First, figure out the possibilities for the cardinality of the equivalence classes. Then count the number of ways of selecting the elements of the classes.

Problem 4. How many ways are there to arrange the letters ALIVE in such a way that the word EVIL is not contained in the resulting word (as a *consecutive* string of letters)?

Problem 5. Say that a full binary tree is a *fully splitting binary tree* if every non-terminal node has exactly 2 successors and all terminal nodes are at the same level—that is, the length of the unique simple path from the root to a terminal vertex is always the same. Prove that, for all non-negative integers n, if T is a fully splitting binary tree of height n, then T has $2^{n+1} - 1$ vertices.

Problem 6. How many solutions are there to the equation

$$x_1 + x_2 + x_3 + x_4 = 17$$

where x_1, x_2, x_3, and x_4 are non-negative integers satisfying $x_1 \geq 2$, $x_2 \leq 14$, and $x_3 \leq 14$.

Problem 7. Let G be a graph whose vertices consist of pairs (a, b) where a and b are non-negative integers less than or equal to 3. Say that there is an edge between (a, b) and (c, d) if either $|a - c| = 1$ and $b = d$ or $|b - d| = 1$ and $a = c$, where $|x|$ denotes the absolute value of x. In other words, two vertices are connected by an edge if they differ by exactly 1 in one coordinate and agree on the other.

1. What is the degree of $(0, 0)$?
2. What is the degree of $(2, 2)$?
3. Define the weight of an edge between (a, b) and (c, d) to be the minimum of the numbers a, b, c, d. What is the weight of a minimal spanning tree for the graph G?

Problem 8. Suppose G is a graph with 17 vertices. Show that if G has at least 9 vertices of degree 9, then G cannot be bipartite.