61 Final Exam

Kendrake Jay Tsui

TOTAL POINTS

75 / 120

QUESTION 1

1 True/False 8 / 10

- **- 2 pts** (1) Incorrect [Correct answer: False]
- **- 2 pts** (2) Incorrect [Correct answer: False]
- **- 2 pts** (3) Incorrect [Correct answer: False]
- ✓ **- 2 pts (4) Incorrect [Correct answer: False]**
- **- 2 pts** (5) Incorrect [Correct answer: True]
- **- 0 pts** Correct

QUESTION 2

2 Sum of power set cardinalities 8 / 10

- **- 0 pts** Correct
- **- 10 pts** not graded
- **- 3 pts** Not enough justification
- **- 1 pts** arithmetic error
- **- 7 pts** major incorrect reasoning

- 2 Point adjustment

sum with k+1 term is incorrect

QUESTION 3

3 Sum of combinations 0 / 10

- **- 0 pts** Correct
- ✓ **- 10 pts Not graded**
- **- 2 pts** missing minor details in reasoning
- **- 5 pts** Incomplete
- **- 7 pts** Major gap missing

QUESTION 4

4 Straight/flush 3 / 10

- **+ 10 pts** Fully/mostly correct
- **+ 0 pts** Skipped
- ✓ **+ 3 pts counted straights**
- **+ 1 pts** [partial credit] Some progress counting straights
- **+ 3 pts** counted flushes
- ✓ **+ 1 pts [partial credit] Some progress counting flushes. (e.g. forgot to include the suits in the count, or counted where order mattered)**
- **+ 2 pts** Counted straight flushes
- **+ 2 pts** Used PIE correctly
- ✓ **- 1 pts Small mistake counting straights**
- **- 1 pts** Small mistake counting flushes
- **- 1 pts** Small mistake counting straight flushes
- **+ 1 pts** [partial credit] Some progress counting straight flushes.
- **- 2 pts** No work shown
- **+ 0 pts** Error when doing unnecessary calculation.

QUESTION 5

5 Integer solutions 0 / 10

- **- 0 pts** Correct with valid work
- **- 2 pts** Minor error (e.g. off by 1 error in the combinations)
- **- 4 pts** Correct calculations for correct cases, but incorrect combination of numbers
- **- 5 pts** Flipped the inequality in the second condition and solved the resulting problem correctly, but in a way that does not scale to the correct problem
- **- 6 pts** Incorrect specification of cases or calculations for said cases
- **- 8 pts** Major errors in setting up cases, counting numbers of solutions in cases, and/or combining the results
- **- 8 pts** A little bit of work
- **- 8 pts** Attempted a constructive count without accounting for changes depending on case
- **- 10 pts** Incorrect with no valid work
- ✓ **- 10 pts Skipped**

QUESTION 6

6 Powerset graph 7 / 10

+ **10 pts** Correct
- **10 pts** Skipped

✓ + **7 pts** **[partial credit] Correct, except empty set was ignored.**

+ **4 pts** G is 3-regular, with some justification (either a picture or an explanation).
+ **6 pts** Correct sum of weights.
+ **4 pts** [partial credit] Small mistake when finding MST
+ **2 pts** [partial credit] Reasonable but incorrect attempt at drawing graph.
+ **1 pts** [partial credit] Some attempt at finding MST. (e.g. Indicating that you know what a spanning tree is.)
+ **3 pts** [partial credit] Correct minimum spanning tree given incorrect picture.
- **1 pts** Wrong definition of d-regular.
- **10 pts** Incorrect

QUESTION 7

7 Sheffer stroke 8 / 10

- **0 pts** Correct
- **10 pts** Skipped
- **9 pts** Only considered particular examples
- **1 pts** Did not justify (3)

- 2 Point adjustment

Justification for (3)?

QUESTION 8

8 Handshake 10 / 10

✓ - **0 pts** **Correct**

- **8 pts** Counted orderings instead of combinations
- **9 pts** Incorrect, unclear what is being counted
- **6 pts** double counted all handshakes
- **10 pts** Did not attempt problem

QUESTION 9

9 Hexagon 10 / 10

✓ - **0 pts** **Correct**

- **10 pts** Skipped
- **2 pts** Right idea, but need to make the structure of your argument more clear. How exactly are you applying the pigeonhole principle?
- **6 pts** Had the idea of using equilateral triangles. But structure of the argument is wrong, or very unclear.
- **6 pts** Tried the "greedy" approach. But didn't justify correctly with pigeonhole.

QUESTION 10

10 Rigid graph 10 / 10

✓ - **0 pts** **Correct**

- **10 pts** Not Graded
- **7 pts** Major flaw in reasoning
- **2 pts** Minor justification needed

QUESTION 11

11 Surjections with small fibers 7 / 10

- **0 pts** Correct

✓ - **2 pts** **Does not correctly account for over-counting when dealing with doubly-covered points of Y**

- **2 pts** Does surjectivity starting from 20 instead of 30
- **2 pts** Does not account for order of assignments when enforcing surjectivity
- **5 pts** Major error in dealing with doubly-covered points of Y
- **8 pts** Builds matchings which may not be valid functions
- **10 pts** Incorrect without a visible way to adapt towards a correct solution
- **10 pts** Interprets the problem as two separate parts
- **10 pts** Skipped

- 1 Point adjustment

Considering permutations of X is necessary in case the "first 20" elements of X do not surject onto Y themselves, but there is a lot of overcounting (corresponding to permutations which shuffle the first 20 amongst themselves).

QUESTION 12

12 Counting rooted graphs **4 / 10**

- **- 0 pts** Correct
- **- 10 pts** skipped
- ✓ **- 6 pts** **Did not take roots into account**
- **- 4 pts** miscounted number of graphs
- **- 5 pts** Count trees on fewer than 3 vertices/2 edges
- **- 7 pts** Counted graphs/rooted trees up to isomorphism, not the number of rooted trees on the given vertices
- **- 8 pts** Miscounted without a clear argument

Name: Kendrake Tsui

Student ID#: 005159259

MATH 61 - FINAL EXAM

0.1. **Instructions.** This is a 180 minute exam. You should feel free to quote any theorems proved in class, as well as anything proved in the homework or discussion section. There are 12 questions—on the exam, you are required to do the first true/false question, and choose 9 of the remaining 11. Only 9 problems other than the true/false question will be graded so *you should indicate which problems you want graded by marking the one you do not want graded with an X*, in the case that you attempt all 6. Each question is worth 10 points. Unless otherwise specified, you are required to justify your answers.

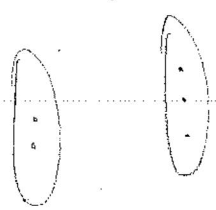
Recall all of our graphs are simple.

Exercise 0.1. Indicate whether the following statements are true or false:

(1) The number of rooted trees on a fixed set of n vertices is n^{n-2}. false
(2) Suppose $|X| = n$ and $|Y| = k$. There are n^k many functions from X to Y. false
(3) If G has no subgraph isomorphic to $K_{3,3}$ or K_5, then G is planar. false
(4) If $G = (V, E)$ is a graph, then the relation R on V, defined by $(x, y) \in R$ if and only if there is a path from x to y in G, is an equivalence relation. true
(5) If $T = (V, E)$ is a rooted tree, then the relation D on V, defined $(x, y) \in D$ if x is a descendent of y or if $x = y$, is a partial order. true

$2^{2-2} = 1$

$3^{3-2} = 3$



$k \cdot k = k^2$

Date: December 9, 2018; Ramsey.

Kendrake Tsui

Exercise 0.2. Recall that for X a set, $\mathcal{P}(X)$ denotes the *power set* of X, the set whose elements are the subsets of X: $\mathcal{P}(X) = \{Y : Y \subseteq X\}$. Show that

$$\sum_{i=1}^{n} |\mathcal{P}(\{1,\ldots,i\})| = 2^{n+1} - 2.$$

Base Case: $n=1 \Rightarrow \sum_{i=1}^{1} |P(\{1\})|$. $P(\{1\}) = \{\}$ and $\{1\}$ so $|P(\{1\})| = 2 = 2^{1+1} - 2$ ✓

Assume true for some k.

Try $k+1$: $\sum_{i=1}^{k+1} |P(\{1,\ldots,k+1\})| = \sum_{i=1}^{k} |P(\{1,\ldots,k\})| + \sum_{k+1}^{k+1} |P(\{k+1\})|$

$$= 2^{k+1} - 2 + \sum_{k+1}^{k+1} |P(\{k+1\})|$$

$$= 2^{k+1} - 2 + 2^{k+1}$$

$$= 2(2^{k+1}) - 2$$

$$= 2^{k+2} - 2 = 2^{(k+1)+1} - 2$$

Since we proved it true for $k+1$, $\sum_{i=1}^{n} |P(\{1,\ldots,i\})| = 2^{n+1} - 2$ holds for $n \geq 1$.

$\square$

Kendrake Tsui

Exercise 0.3. Show that, for all natural numbers n,

$$2^n = \binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{n} = \sum_{i=0}^{n} \binom{n}{i}.$$

Note: Both algebraic and combinatorial proofs are possible.

$$\sum_{i=0}^{n} \binom{n}{i} = \frac{n!}{(n-0)!0!} + \frac{n!}{(n-1)!1!} + \frac{n!}{(n-2)!2!} + \ldots + \frac{n!}{(n-n)!n!}$$

Kendrake Tsui

Exercise 0.4. There are $\binom{52}{5}$ many 5-card hands from a standard 52 card deck. How many of them are either a *straight* or a *flush*? (A *straight* is a sequence of 5 cards whose values are in order, the ace is high (and not low); a *flush* is 5 cards of the same suit).

Straight: $\binom{28}{1}\binom{4}{1}\binom{4}{1}\binom{4}{1}\binom{4}{1} + \binom{4}{1}\binom{4}{1}\binom{4}{1}\binom{4}{1}\binom{4}{1}$

Pick 2-8 first   Pick a 9 first

Flush: $\binom{52}{1}\binom{12}{1}\binom{11}{1}\binom{10}{1}\binom{9}{1}$

Straight or Flush: $\binom{28}{1}\binom{4}{1}\binom{4}{1}\binom{4}{1}\binom{4}{1} + \binom{4}{1}^5 + 52(12)(11)(10)(9)$

$= 28(4)^4 + (4)^5 + 52\left(\frac{12!}{8!}\right)$

Kendrake Tsui

Exercise 0.5. How many integer solutions are there to the equation

$$x_1 + x_2 + x_3 = 37$$

subject to the constraints that $3 \leq x_1 < 6$, $4 \leq x_2$, and $0 \leq x_3 < 37$.

Total: $\binom{37+3-1}{3-1} = \binom{39}{2}$

$X_1 \geq 3$: $\binom{37+3-3-1}{3-1} = \binom{36}{2}$

$X_1 \geq 7$: $\binom{37+3-7-1}{3-1} = \binom{32}{2}$ ← bad

$X_2 \geq 4$: $\binom{37+3-4-1}{3-1} = \binom{35}{2}$

$X_1 \geq 3$ and $X_2 \geq 4$: $\binom{37+3-3-4-1}{3-1} = \binom{32}{2}$

$X_1 \geq 7$ and $X_2 \geq 4$: $\binom{28}{2}$

$$\underbrace{\binom{36}{2}}_{X_1 \geq 3} + \underbrace{\binom{35}{2}}_{X_2 \geq 4} - \underbrace{\binom{32}{2}}_{X_1 \geq 3,\ X_2 \geq 4} - \underbrace{\binom{32}{2}}_{X_1 \geq 7}$$

$$\underbrace{\binom{39}{2}}_{\text{total}} - \underbrace{\binom{32}{2}}_{X_1 > 6} - \underbrace{\left[\binom{39}{2} - \binom{35}{2}\right]}_{X_2 < 4} - \underbrace{\left[\binom{39}{2} - \binom{36}{2}\right]}_{X_1 < 3} + \underbrace{\left[\binom{39}{2} - \binom{32}{2}\right]}_{X_1 < 3 \text{ and } X_2 < 4} + \underbrace{\left[\binom{39}{2} - \binom{28}{2}\right]}_{X_1 > 6 \text{ and } X_2 < 4}$$

Kendrake Tsui

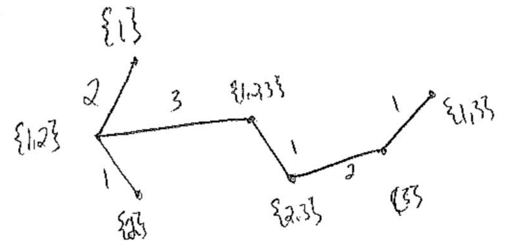
Exercise 0.6. Consider a weighted graph $G = (V, E)$ where $V = \mathcal{P}(\{1,2,3\})$ and v and w are connected by an edge if and only if $|v \Delta w| = 1$, in which case this edge is given weight corresonding to the unique element of $v \Delta w$. In other words, the vertices of G are the subsets of $\{1,2,3\}$ and the edges are pairs of sets where one has exactly one more element than the other. For example, $\{1\}$ and $\{1,2\}$ are connected by an edge, and the weight of this edge is 2.

(1) Is there some d so that G is d-regular?
(2) What is the sum of the weights in a minimal spanning tree?

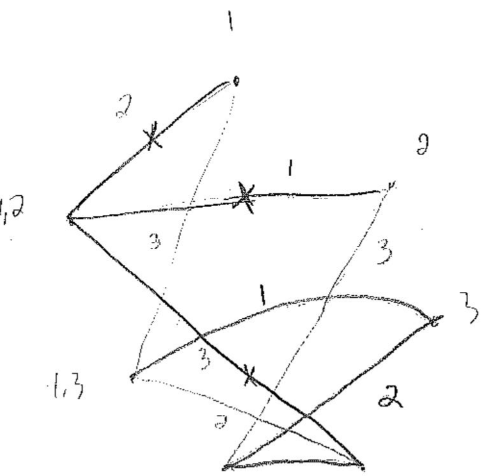
1) $\{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}$
2 2 2 3 3 3 3

There is no d so that G is d-regular because subsets v_1 of size one have two distinct w so that $|v_1 \Delta w| = 2$, subsets v_2 of size 2 have 3 different w such that $|v_2 \Delta w| = 1$ and subsets V_3 of size V_3 have 3 distinct w such that $|v_3 \Delta w| = 1$. Since all edges are in the graph if they satisfy $|v \Delta w| = 1$, and $|v_1 \Delta w| \neq |v_2 \Delta w| \neq |v_3 \Delta w|$, there is no d so G is d-regular.

2)



$\boxed{\text{Sum} = 10}$



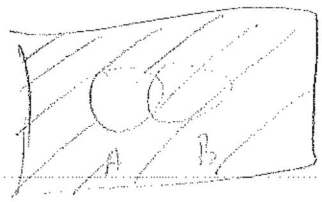
Kendrake Tsui

Exercise 0.7. Suppose X is the universal set and for subsets $A \subseteq X$, we write $\overline{A}$ to denote the complement of A in X. The *Sheffer stroke* $\uparrow$ is a single operation on sets defined by $A \uparrow B$ is the complement of $A \cap B$. This has the remarkable property that every other operation on sets may be defined in terms of it.

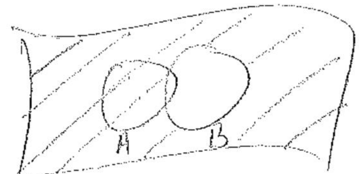
(1) Show $\overline{A} = (A \uparrow A)$.
(2) Show $A \cup B = (A \uparrow A) \uparrow (B \uparrow B)$
(3) Find a formula for $A \cap B$ using only parentheses and $\uparrow$.

1) Suppose $(A \uparrow A)$. $A \cap A = A$ because all the elements in A are contained in A. Therefore $A \uparrow A$ is the complement to $A \cap A$ which is just the complement to A. Therefore $\overline{A} = (A \uparrow A)$.

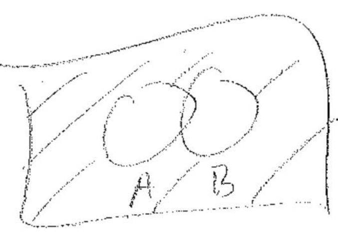
2) Given $(A \uparrow A)$ and $(B \uparrow B)$, we know from what we proved in the problem above that $(A \uparrow A) = \overline{A}$ and $(B \uparrow B) = \overline{B}$. We can rewrite $(A \uparrow A) \uparrow (B \uparrow B)$ as $\overline{A} \uparrow \overline{B}$ which is $\overline{(\overline{A} \cap \overline{B})}$. $\overline{A} \cap \overline{B}$ is everything except for $A \cup B$ because the only elements $\overline{A}$ and $\overline{B}$ share are the elements outside both A and B. Therefore the complement to $\overline{A} \cap \overline{B}$ is everything in both A and B which is $A \cup B$. So $A \cup B = (A \uparrow A) \uparrow (B \uparrow B)$



$\overline{A}$

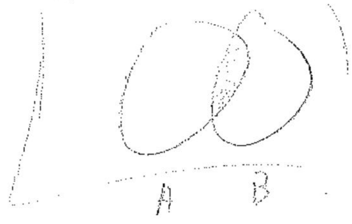


$\overline{B}$



$\overline{A} \cap \overline{B}$

3) $(A \uparrow B) \uparrow (A \uparrow B)$



Kendrake Tsui

Exercise 0.8. At the end of finals week there will be a party with 15 guests.

(1) If each person shakes every other person's hand exactly once, how many handshakes have taken place?

(2) Suppose that 7 of the guests have the same cold, so they don't want to shake the hands of the guests who are not sick, but they don't mind shaking each others' hands. If all the sick guests shake hands and all the not-sick guests shake hands, how many hand shakes have taken place?

1) $\frac{(15)(14)}{2}$ handshakes $\boxed{=105}$

14 13 12 11 10 9 8 7 6 5 4 3 2 1 _

$15 \times 7 = 105$

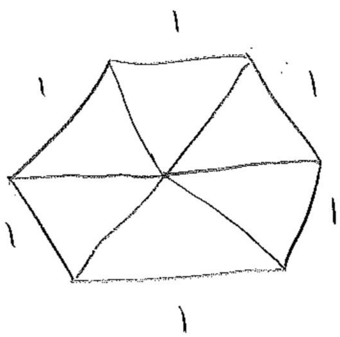
K_{15} has $\frac{(15)(14)}{2}$ edges

2) $K_7 + K_8$

$\frac{(7)(6)}{2} + \frac{(8)(7)}{2} = 21 + 28 = \boxed{49}$

Kendrake Tsui

Exercise 0.9. Suppose there are 7 points in the interior of a regular hexagon (i.e. all interior angles are the same) of side length 1. Show that at least two of the points are within distance 1 of one another.



$$\begin{array}{r}120\\ 6\\ \hline 720\end{array}$$

Subdivide the hexagon into six equal triangles by connecting opposite vertices with each other. One one these triangles must have at least 2 points inside of it or on its boundary. These triangles are all equilateral so they all have side length one. Therefore there are at least two points that are within distance one of each other.

□

Kendrake Tsui

Exercise 0.10. Recall that a graph G is called *rigid* if the only autorphism of G is the identity function - i.e. the function that sends $v \mapsto v$ and $e \mapsto e$ for every vertex and edge of G respectively. Is there a rigid graph with exactly 3 vertices? Give an example or show there is no such graph.

There is no rigid graph with only three vertices.
Given G_1 and G_2, both with no edges we can take a bijection on the vertices of G_1 onto G_2. Any of the vertices can go to any of the vertices in G_2 because each vertex has degree 0, therefore the graph is not rigid.

Given G_3 and G_4, both with one edge, we can take a bijection such that the node with degree 0 in G_3 goes to the node of degree 0 in G_4. We can then send the remaining two nodes of G_3 to any of the two nodes in G_4 since the remaining nodes have degree one. This is not rigid because we could send the first node of degree 1 to any of two choices in G_4.

Given G_5 and G_6, both of which have two edges, we can take a bijection from G_5 to G_6 such that the node of degree 2 in G_5 goes to the node of degree 2 in G_6. We can then map the remaing two nodes of degree one in G_5 to the two nodes of degree one in G_6. We have two choices on where to send the first node of degree one in G_5 so it is not rigid.

Finally given G_7 and G_8, both which have 3 edges, we can take a bijection that maps any of the nodes of G_7 to any on G_8 since they all have degree 2, therefore it is not rigid.

Since none of the cases with a graph with 3 vertices are rigid, there is no rigid graph on 3 vertices. $\square$

Kendrake Tsui

Exercise 0.11. Suppose X is a set with 30 elements and Y is a set with 20 elements. How many functions $f : X \to Y$ are there satisfying both of the following properties (at the same time):

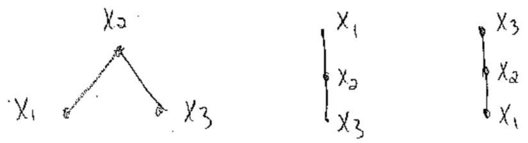
(1) $f : X \to Y$ is onto (i.e. surjective)
(2) For every $y \in Y$, $|\{x \in X : f(x) = y\}| \leq 2$.

$20!\left(\frac{20!}{10!}\right)(30!)$

The first 20 elements in X must hit each element in Y so there are $20!$ ways to do that. With the remaining ten elements in X, they can only hit one of the elements in Y one more time because $\forall y \in Y, |\{x \in X : f(x) = y\}| \leq 2$. Therefore the 21^{st} element in X can hit any of the 20 elements in Y again but the 22^{nd} element in X only has 19 choices remaining. We then multiply by $|X|! = 30!$ because there are $30!$ ways to permute the elements of X. ☐

Kendrake Tsui

Exercise 0.12. How many rooted trees are there on the vertices $\{x_1, x_2, x_3\}$? Are there more or fewer than the total number of graphs on these vertices?



There are three rooted trees on the vertices $\{x_1, x_2, x_3\}$

There are fewer than the total number of graphs on these vertices because there are $2^{\binom{3}{2}} = 2^3 = 8$ possible graphs. These include the complete graph K_3 and also the graphs which are not connected. K_3 cannot be a tree because it is cyclic and trees must also be connected. Therefore there are more graphs than rooted trees. $\square$