61 Final Exam

Nathan George Midkiff

TOTAL POINTS

100 / 120

QUESTION 1

1 True/False 10 / 10

- **- 2 pts** (1) Incorrect [Correct answer: False]
- **- 2 pts** (2) Incorrect [Correct answer: False]
- **- 2 pts** (3) Incorrect [Correct answer: False]
- **- 2 pts** (4) Incorrect [Correct answer: False]
- **- 2 pts** (5) Incorrect [Correct answer: True]
- ✓ **- 0 pts Correct**

QUESTION 2

2 Sum of power set cardinalities 10 / 10

- ✓ **- 0 pts Correct**
- **- 10 pts** not graded
- **- 3 pts** Not enough justification
- **- 1 pts** arithmetic error
- **- 7 pts** major incorrect reasoning

QUESTION 3

3 Sum of combinations 10 / 10

- ✓ **- 0 pts Correct**
- **- 10 pts** Not graded
- **- 2 pts** missing minor details in reasoning
- **- 5 pts** Incomplete
- **- 7 pts** Major gap missing

QUESTION 4

4 Straight/flush 10 / 10

- ✓ **+ 10 pts Fully/mostly correct**
- **+ 0 pts** Skipped
- **+ 3 pts** counted straights
- **+ 1 pts** [partial credit] Some progress counting straights
- **+ 3 pts** counted flushes
- **+ 1 pts** [partial credit] Some progress counting flushes. (e.g. forgot to include the suits in the count, or counted where order mattered)
- **+ 2 pts** Counted straight flushes
- **+ 2 pts** Used PIE correctly
- **- 1 pts** Small mistake counting straights
- **- 1 pts** Small mistake counting flushes
- **- 1 pts** Small mistake counting straight flushes
- **+ 1 pts** [partial credit] Some progress counting straight flushes.
- **- 2 pts** No work shown
- **+ 0 pts** Error when doing unnecessary calculation.

QUESTION 5

5 Integer solutions 0 / 10

- **- 0 pts** Correct with valid work
- **- 2 pts** Minor error (e.g. off by 1 error in the combinations)
- **- 4 pts** Correct calculations for correct cases, but incorrect combination of numbers
- **- 5 pts** Flipped the inequality in the second condition and solved the resulting problem correctly, but in a way that does not scale to the correct problem
- **- 6 pts** Incorrect specification of cases or calculations for said cases
- **- 8 pts** Major errors in setting up cases, counting numbers of solutions in cases, and/or combining the results
- **- 8 pts** A little bit of work
- **- 8 pts** Attempted a constructive count without accounting for changes depending on case
- **- 10 pts** Incorrect with no valid work
- ✓ **- 10 pts Skipped**

QUESTION 6

6 Powerset graph 10 / 10

- ✓ **+ 10 pts Correct**

- **10 pts** Skipped
+ **7 pts** [partial credit] Correct, except empty set was ignored.
+ **4 pts** G is 3-regular, with some justification (either a picture or an explanation).
+ **6 pts** Correct sum of weights.
+ **4 pts** [partial credit] Small mistake when finding MST
+ **2 pts** [partial credit] Reasonable but incorrect attempt at drawing graph.
+ **1 pts** [partial credit] Some attempt at finding MST. (e.g. Indicating that you know what a spanning tree is.)
+ **3 pts** [partial credit] Correct minimum spanning tree given incorrect picture.
- **1 pts** Wrong definition of d-regular.
- **10 pts** Incorrect

QUESTION 7

7 Sheffer stroke 10 / 10

✓ **- 0 pts** Correct
- **10 pts** Skipped
- **9 pts** Only considered particular examples
- **1 pts** Did not justify (3)

QUESTION 8

8 Handshake 10 / 10

✓ **- 0 pts** Correct
- **8 pts** Counted orderings instead of combinations
- **9 pts** Incorrect, unclear what is being counted
- **6 pts** double counted all handshakes
- **10 pts** Did not attempt problem

QUESTION 9

9 Hexagon 10 / 10

✓ **- 0 pts** Correct
- **10 pts** Skipped
- **2 pts** Right idea, but need to make the structure of your argument more clear. How exactly are you applying the pigeonhole principle?
- **6 pts** Had the idea of using equilateral triangles. But structure of the argument is wrong, or very unclear.
- **6 pts** Tried the "greedy" approach. But didn't justify correctly with pigeonhole.

QUESTION 10

10 Rigid graph 10 / 10

✓ **- 0 pts** Correct
- **10 pts** Not Graded
- **7 pts** Major flaw in reasoning
- **2 pts** Minor justification needed

QUESTION 11

11 Surjections with small fibers 0 / 10

- **0 pts** Correct
- **2 pts** Does not correctly account for over-counting when dealing with doubly-covered points of Y
- **2 pts** Does surjectivity starting from 20 instead of 30
- **2 pts** Does not account for order of assignments when enforcing surjectivity
- **5 pts** Major error in dealing with doubly-covered points of Y
- **8 pts** Builds matchings which may not be valid functions
- **10 pts** Incorrect without a visible way to adapt towards a correct solution
- **10 pts** Interprets the problem as two separate parts

✓ **- 10 pts** **Skipped**

QUESTION 12

12 Counting rooted graphs 10 / 10

✓ **- 0 pts** Correct
- **10 pts** skipped
- **6 pts** Did not take roots into account
- **4 pts** miscounted number of graphs
- **5 pts** Count trees on fewer than 3 vertices/2 edges
- **7 pts** Counted graphs/rooted trees up to isomorphism, not the number of rooted trees on the given vertices
- **8 pts** Miscounted without a clear argument

gradescope

Name: Nathan Midkiff

Student ID#: 804979581

MATH 61 - FINAL EXAM

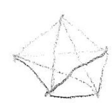
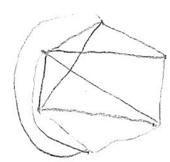
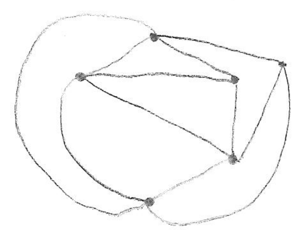
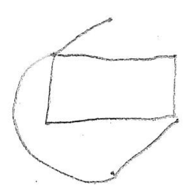
0.1. **Instructions.** This is a 180 minute exam. You should feel free to quote any theorems proved in class, as well as anything proved in the homework or discussion section. There are 12 questions—on the exam, you are required to do the first true/false question, and choose 9 of the remaining 11. Only 9 problems other than the true/false question will be graded so *you should indicate which problems you want graded by marking the one you do not want graded with an X*, in the case that you attempt all 6. Each question is worth 10 points. Unless otherwise specified, you are required to justify your answers.

Recall all of our graphs are simple.

Exercise 0.1. Indicate whether the following statements are true or false:

(1) The number of rooted trees on a fixed set of n vertices is n^{n-2}.
(2) Suppose $|X| = n$ and $|Y| = k$. There are n^k many functions from X to Y.
(3) If G has no subgraph isomorphic to $K_{3,3}$ or K_5, then G is planar.
(4) If $G = (V, E)$ is a graph, then the relation R on V, defined by $(x, y) \in R$ if and only if there is a path from x to y in G, is an equivalence relation.
(5) If $T = (V, E)$ is a rooted tree, then the relation D on V, defined $(x, y) \in D$ if x is a descendent of y or if $x = y$, is a partial order.

1) False
2) False
3) False
4) False, not reflexive
5) True

Date: December 9, 2018; Ramsey.

Nathan Midkiff
804979581

Exercise 0.2. Recall that for X a set, $\mathcal{P}(X)$ denotes the *power set* of X, the set whose elements are the subsets of X: $\mathcal{P}(X) = \{Y : Y \subseteq X\}$. Show that

$$\sum_{i=1}^{n} |\mathcal{P}(\{1,\ldots,i\})| = 2^{n+1} - 2.$$

Basis: $i=1$

LHS: $\sum_{i=1}^{1} |P(\{1\})| = |P(\{1\})| = |\{\emptyset, \{1\}\}| = 2$

RHS: $2^{1+1} - 2 = 4 - 2 = 2$ ✓

Inductive step: Assume $\sum_{i=1}^{k} |P(\{1,\ldots,i\})| = 2^{k+1} - 2$, show $\sum_{i=1}^{k+1} |P(\{1,\ldots,i\})| = 2^{(k+1)+1} - 2$

$\sum_{i=1}^{k+1} |P(\{1,\ldots,i\})| = \sum_{i=1}^{k} |P(\{1,\ldots,i\})| + |P(\{1,\ldots,k+1\})|$

$= 2^{k+1} - 2 + |P(\{1,\ldots,k+1\})|$, by inductive hypothesis

$|P(\{1,\ldots,k+1\})| = 2^{k+1}$ because for each element, i, of $\{1,\ldots,k+1\}$, there are two choices, $i \in Y$ or $i \notin Y$. Because there are $k+1$ elements i, and each choice is independent of one another, there are 2^{k+1} total choices.

$\Rightarrow 2^{k+1} - 2 + |P(\{1,\ldots,k+1\})| = 2^{k+1} - 2 + 2^{k+1} = 2 \cdot 2^{k+1} - 2 = 2^{(k+1)+1} - 2$

$\Rightarrow \sum_{i=1}^{k+1} |P(\{1,\ldots,i\})| = 2^{(k+1)+1} - 2$ ✓

So, by induction,

$\sum_{i=1}^{n} |P(\{1,\ldots,i\})| = 2^{n+1} - 2$, for $n \geq 1$

Nathan Midkiff
304979581

Exercise 0.3. Show that, for all natural numbers n,

$$2^n = \binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{n} = \sum_{i=0}^{n} \binom{n}{i}.$$

Note: Both algebraic and combinatorial proofs are possible.

If you have an n element set, there are 2^n possible subsets, because for each element, it is either in or not in the subset.

On the other hand, any subset of a set of n elements has some size $i \in \{0, \ldots, n\}$. So the number of possible subsets is the sum of the number of possible subsets of size i, for all $i \in \{0, \ldots, n\}$. The number of possible subsets of size i is $\binom{n}{i}$, so the total number of subsets is $\sum_{i=0}^{n} \binom{n}{i}$

So $2^n = \sum_{i=0}^{n} \binom{n}{i}$, $n \geq 0$

Nathan Midkiff
304979581

Exercise 0.4. There are $\binom{52}{5}$ many 5-card hands from a standard 52 card deck. How many of them are either a *straight* or a *flush*? (A *straight* is a sequence of 5 cards whose values are in order, the ace is high (and not low); a *flush* is 5 cards of the same suit).

\# straights: 2, 3, 4, 5, 6 | 5, 6, 7, 8, 9 | 8, 9, 10, J, Q
3, 4, 5, 6, 7 | 6, 7, 8, 9, 10 | 9, 10, J, Q, K
4, 5, 6, 7, 8 | 7, 8, 9, 10, J | 10, J, Q, K, A

Each card: 4 suits

$\Rightarrow 9 \cdot 4^5$

\# flushes: $\underset{\text{suit}}{4} \cdot \underset{\text{cards in suit}}{\binom{13}{5}} = 4 \cdot \frac{13!}{5!8!}$

\# straight flushes: $9 \cdot 4 = 36$

total $= \boxed{9 \cdot 4^5 + \frac{4 \cdot 13!}{5!8!} - 36}$

A1 B1 C1
A2 B2 C2

Nathan Nidkiff
904979581

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Exercise 0.5. How many integer solutions are there to the equation

$$x_1 + x_2 + x_3 = 37$$

subject to the constraints that $3 \leq x_1 < 6$, $4 \leq x_2$, and $0 \leq x_3 < 37$.

0, 0, 3	1, 0, 2	2, 1 0
0, 1, 2	1, 1, 1	3, 0, 0
0, 2, 1	1, 2, 1	
0, 3, 0	2, 0, 1	

$$\sum_{i=0}^{x} \binom{n+v-1-x}{v-1}$$

Nathan Midkiff

304979581

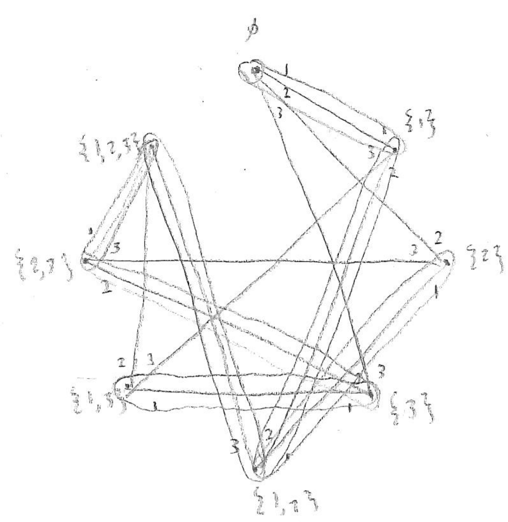
$V \cup V - V \cap V$

Exercise 0.6. Consider a weighted graph $G = (V, E)$ where $V = \mathcal{P}(\{1,2,3\})$ and v and w are connected by an edge if and only if $|v \Delta w| = 1$, in which case this edge is given weight corresonding to the unique element of $v \Delta w$. In other words, the vertices of G are the subsets of $\{1,2,3\}$ and the edges are pairs of sets where one has exactly one more element than the other. For example, $\{1\}$ and $\{1,2\}$ are connected by an edge, and the weight of this edge is 2.

(1) Is there some d so that G is d-regular?
(2) What is the sum of the weights in a minimal spanning tree?

1) G is 3-regular

2) $1 + 2 + 1 + 3 + 1 + 2 + 1 = 11$



Nathan Midkiff

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Exercise 0.7. Suppose X is the universal set and for subsets $A \subseteq X$, we write $\overline{A}$ to denote the complement of A in X. The *Sheffer stroke* $\uparrow$ is a single operation on sets defined by $A \uparrow B$ is the complement of $A \cap B$. This has the remarkable property that every other operation on sets may be defined in terms of it.

(1) Show $\overline{A} = (A \uparrow A)$.
(2) Show $A \cup B = (A \uparrow A) \uparrow (B \uparrow B)$
(3) Find a formula for $A \cap B$ using only parentheses and $\uparrow$.

1) If $b \in \bar{A}$, then $b \notin A$ by definition. Because $A = A \cap A$, $b \notin A \cap A$, so $b \in \overline{A \cap A}$

so $b \in (A \uparrow A)$, so $\bar{A} \subseteq (A \uparrow A)$

If $c \in (A \uparrow A)$, $c \in \overline{A \cap A}$, so $c \notin A \cap A$, so $c \notin A$, so $c \in \bar{A}$, so $(A \uparrow A) \subseteq \bar{A}$.

$\Rightarrow \bar{A} = (A \uparrow A)$

2) $(A \uparrow A) \uparrow (B \uparrow B) = \bar{A} \uparrow \bar{B} = \overline{(\bar{A} \cap \bar{B})}$

If $a \in A \cup B$ either

i) $a \in A$, so $a \notin \bar{A}$, so $a \notin \bar{A} \cap \bar{B}$, so $a \in \overline{(\bar{A} \cap \bar{B})}$

or ii) $a \in B$, so $a \notin \bar{B}$, so $b \notin \bar{A} \cap \bar{B}$, so $a \in \overline{(\bar{A} \cap \bar{B})}$

In either case, $a \in \overline{(\bar{A} \cap \bar{B})}$, so $A \cup B \subseteq \overline{(\bar{A} \cap \bar{B})}$

If $b \in \overline{(\bar{A} \cap \bar{B})}$, $b \notin (\bar{A} \cap \bar{B})$, which means either

i) $b \notin \bar{A}$, so $b \in A$

or ii) $b \notin \bar{B}$, so $b \in B$

so $b \in A$ or $b \in B$, so $b \in A \cup B$, so $\overline{(\bar{A} \cap \bar{B})} \subseteq A \cup B$

$\Rightarrow A \cup B = \overline{(\bar{A} \cap \bar{B})}$

$\Rightarrow A \cup B = (A \uparrow A) \uparrow (B \uparrow B)$

3) $A \cap B = (A \uparrow B) \uparrow (A \uparrow B)$

Nathan Midkiff
80497958l

Exercise 0.8. At the end of finals week there will be a party with 15 guests.

(1) If each person shakes every other person's hand exactly once, how many handshakes have taken place?
(2) Suppose that 7 of the guests have the same cold, so they don't want to shake the hands of the guests who are not sick, but they don't mind shaking each others' hands. If all the sick guests shake hands and all the not-sick guests shake hands, how many hand shakes have taken place?

$\overset{3}{15} \times 7 = 105$

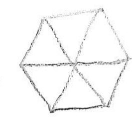
1) $\binom{15}{2} = \frac{15 \cdot 14}{2} = 105$

2) $\binom{7}{2} + \binom{8}{2} = \frac{7 \cdot 6}{2} + \frac{8 \cdot 7}{2} = 21 + 28 = 49$

Nathan Midkiff
804979581

Exercise 0.9. Suppose there are 7 points in the interior of a regular hexagon (i.e. all interior angles are the same) of side length 1. Show that at least two of the points are within distance $\cancel{\frac{1}{2}}$ of one another.

1

Partition the hexagon into 6 equilateral triangles of side length 1 => 

With 7 points inside 6 triangles, at least 1 triangle must have at least two points, by the pigeonhole principle. Because these two points are inside an equilateral triangle of side 1, they are within a distance 1 of each other. So at least two points are within distance 1 of each other.

Nathan Midkiff
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Exercise 0.10. Recall that a graph G is called *rigid* if the only autorphism of G is the identity function - i.e. the function that sends $v \mapsto v$ and $e \mapsto e$ for every vertex and edge of G respectively. Is there a rigid graph with exactly 3 vertices? Give an example or show there is no such graph.

No

G can have e edges, $e \in \{0, \ldots, \binom{3}{2}\} \Rightarrow e \in \{0,1,2,3\}$

Let $G = \{V, E\}$, $V = \{a, b, c\}$

If $|E| = 0$, the isomorphic function is

$f(a) = b$, $f(b) = c$, $f(c) = a$

this is bijective, so there is an autorphism that is not the identity function.

If $|E| = 1$, then the graph has one vertex with degree 0, and two with degree 1.

The autorphic function maps one vertex of degree 1 to the other vertex of degree 1, and the vertex of degree 0 to itself. Also, $f(\{v_1, v_2\}) = \{f(v_1), f(v_2)\}$ for all $\{v_1, v_2\} \in E$.

this is bijective on edges and vertices, so there is an autorphism that is not the identity function.

If $|E| = 2$, then the graph has one vertex with degree 2, and two with degree 1.

The autorphic function maps one vertex of degree 1 to the other vertex of degree 1, and the vertex of degree 0 to itself. Also, $f(\{v_1, v_2\}) = \{f(v_1), f(v_2)\}$ for all $\{v_1, v_2\} \in E$

this is bijective, so there is an autorphism that is not the identity function

If $|E| = 3$, then the autorphic function is

$f(a) = b$, $f(b) = c$, $f(c) = a$, $f(\{a,b\}) = \{b,c\}$, $f(\{a,c\}) = \{b,a\}$. $f(\{b,c\}) = \{c,a\}$

this is bijective, so there is an autorphism that is not the identity function.

So, for all graphs G, there exists an autorphism besides the identity function so no graphs with 3 vertices are rigid.

Nathan Midkiff
804979581

Exercise 0.11. Suppose X is a set with 30 elements and Y is a set with 20 elements. How many functions $f : X \to Y$ are there satisfying both of the following properties (at the same time):

(1) $f : X \to Y$ is onto (i.e. surjective)
(2) For every $y \in Y$, $|\{x \in X : f(x) = y\}| \leq 2$.

Nathan Midkiff

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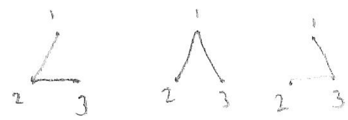
Exercise 0.12. How many rooted trees are there on the vertices $\{x_1, x_2, x_3\}$? Are there more or fewer than the total number of graphs on these vertices?

trees:



rooted trees $= 3 \cdot 3 = \boxed{9}$

total # graphs $= 2^{\binom{3}{2}} = 2^3 = 8$

$\boxed{\text{more}}$