61 Midterm 2

Michael Wu

TOTAL POINTS

57 / 70

QUESTION 1

1 Exercise 1 **8 / 10**

- **- 2 pts** (1) Incorrect [correct response T]
- **- 2 pts** (2) Incorrect [correct response F]
- ✓ **- 2 pts (3) Incorrect [correct response T]**
- **- 2 pts** (4) Incorrect [correct response T]
- **- 2 pts** (5) Incorrect [correct response F]
- **- 0 pts** All correct

(3) Each of n elements in X has n choices for where it goes.

QUESTION 2

2 Exercise 2 **0 / 10**

- ✓ **- 5 pts (1) Incorrect**
- **- 1 pts** (2) Incorrect: off-by-one error.
- **- 2 pts** (2) Incorrect: gave answer m choose n instead of (m choose n) * n!.
- ✓ **- 5 pts (2) Incorrect: other reason**
- **- 0 pts** Both correct
- **- 10 pts** Skipped

(1) Defining a function does not care about an "order of assignment" of values. Also, independent choices lead to multiplication rather than addition.

QUESTION 3

3 Exercise 3 **10 / 10**

- ✓ **- 0 pts Correct**
- **- 3 pts** Miscounted letters/repeats (e.g. answered 10!/2!)
- **- 10 pts** Not graded
- **- 8 pts** Only recognized 10! permutations without considering repeats. (Or dealt with repeats incorrectly.)

QUESTION 4

4 Exercise 4 **10 / 10**

- ✓ **- 0 pts Correct**
- **- 10 pts** Skipped
- **- 7 pts** No further than expanding binomials (correctly)
- **- 2 pts** Small algebra error
- **- 4 pts** Made progress, but substantial algebra error or didn't finish
- **- 8 pts** Showed some indication of what the individual terms mean combinatorially, but not how they're related
- **- 0 pts** Click here to replace this description.

QUESTION 5

5 Exercise 5 **10 / 10**

- ✓ **- 0 pts Correct**
- **- 10 pts** Incorrect
- **- 3 pts** Apply Pigeonhole Incorrectly
- **- 8 pts** Didn't apply Pigeonhole
- **- 10 pts** Not Graded
- **- 5 pts** Flawed argument, did not consider birthdays on same day

QUESTION 6

6 Exercise 6 **10 / 10**

- ✓ **- 0 pts Correct**
- **- 10 pts** skipped
- **- 3 pts** Algebra mistakes in solving for coefficients
- **- 5 pts** Incorrect auxiliary polynomial
- **- 9 pts** Incorrect, no attempt at application of method
- **- 6 pts** No solution after finding aux polynomial and root
- **- 5 pts** Incorrect solution form
- **- 5 pts** Incorrect root for aux poly

QUESTION 7

7 Exercise 7 9 / 10

✓ **- 0 pts** Correct

- 10 pts Skipped

✓ **- 1 pts** Correct answer, but only justified by a picture

- 1 pts Correct answer, but missing justification

- 4 pts Omitted empty graph

- 9 pts Incorrect, with no clear justification

- 8 pts Miscounted because had edges not connected to the given vertices

- 8 pts Miscounted because counted vertices that were connected to other vertices

- 5 pts Did not include the empty graph in the count

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MATH 61 - MIDTERM EXAM 1

0.1. **Instructions.** This is a 50 minute exam. You should feel free to quote any theorems proved in class, as well as anything proved in the homework or discussion section. There are 7 questions—on the real exam, you are required to do the first true/false question, and choose 5 of the remaining 6. Only 5 problems other than the true/false question will be graded so *you should indicate which problems you want graded by marking the one you do not want graded with an X*, in the case that you attempt all 6. Each question is worth 10 points. Unless otherwise specified, you are required to justify your answers.

Exercise 0.1. Indicate whether the following statements are true or false. You do not need to justify your answer.

(1) Suppose X and Y are finite sets with $|X|$ and $|Y|$ even. Then $|X \Delta Y|$ is even.

(2) If $0 \leq k \leq n$,

$$\binom{n}{k} = \binom{n-k}{k}.$$

$\binom{n}{k} = \binom{n}{n-k}$

(3) If $|X| = n$, there are n^n functions from X to X.

(4) There are $2^n - 1$ sequences of 0s and 1s of length n with at least one 0.

(5) Any sequence $\{a_n\}$ satisfying the recurrence relation

$$a_{n+2} = 4a_{n+1} + 2a_n$$

must have only even terms.

(1) true

(2) false

(3) false n^2

(4) true

(5) false $a_1 = 1$ ✓ $a_0 = 0$ ✓

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X

Exercise 0.2. Suppose X and Y are finite sets with $n = |X| < |Y| = m$.

(1) Show there are $n!$ functions from X to X that are both injective and surjective.
relations 1:1 onto

(2) How many injective functions are there from X to Y?

(1) $f(x) = y \quad x, y \in X$

first $x \to n$ values to choose from

first $y \to n$ values to choose from

2nd $x \to n-1$ "

2nd $y \to n-1$ "

$n^2 + (n-1)^2 + (n-2)^2 + \ldots + 1$

Exercise 0.3. How many words can be obtained by rearranging the letters in

$$CALIFORNIA$$

(**Note**: they do not need to be real words and you can leave your answer in combinatorial notation)?

$$\frac{10!}{2!\cdot 2!}$$

Exercise 0.4. Show that if $2 \le k \le n$, then

$$\binom{n}{k} + 2\binom{n}{k-1} + \binom{n}{k-2} = \binom{n+2}{k}.$$

$$\frac{n!}{(n-k)!\,k!} + \frac{2n!}{(n-k+1)!\,(k-1)!} + \frac{n!}{(n-k+2)!\,(k-2)!}$$

$$= \frac{n!\,(n-k+1)(n-k+2) + 2n!\,(n-k+2)k + n!\,(k-1)k}{(n-k+2)!\,k!}$$

$$= \frac{n!\,\big((n-k+1)(n-k+2) + 2(n-k+2)k + (k-1)k\big)}{(n-k+2)!\,k!}$$

$$= \frac{n!\,(n^2 \cancel{-nk} + 2n \cancel{-nk} + k^2 \cancel{-2k} + n \cancel{-k} + 2 + 2nk - 2k^2 + \cancel{4k} + k^2 \cancel{-k})}{(n-k+2)!\,k!}$$

$$= \frac{n!\,(n^2 + 2n + n + 2k^2 - 2k^2 + 2)}{(n-k+2)!\,k!}$$

$$= \frac{n!\,(n^2+3n+2)}{(n-k+2)!\,k!}$$

$$= \frac{n!\,(n+2)(n+1)}{(n+2-k)!\,k!}$$

$$= \frac{(n+2)!}{(n+2-k)!\,k!} = \binom{n+2}{k}$$

Exercise 0.5. Let's pretend that there are no leap years so every year has 365 days. Show that if there are 185 students in our class, then there are two students who have either the same birthday or have consecutive birthdays.

let $B = \{b_1, b_2, \dots, b_{185}\}$ be the set of all birthdays of 185 students.

let $T = \{b_i+1, b_2+1, \dots, b_{185}+1\}$.

then $|B \cup T| = 185 \times 2 = 370$

$f: (B \cup T) \rightarrow$ (days in year)
(cardinality 370) (cardinality 365)

there are only 365 days in a year.

By the pigeonhole principle, at least 2 elements in set $(B \cup T)$ must be the same, since the elements represent dates. Thus, either 2 birthdays are consecutive or they are the same.

In otherwords, some $b_i = b_j + 1$ OR some $b_i = b_j$ for $i \neq j$.

In the case where some $b_i + 1 = b_j + 1$, $b_i = b_j$.

Exercise 0.6. Find the solution to the recurrence relation

$$a_n = 6a_{n-1} - 9a_{n-2},$$

subject to the initial conditions $a_0 = 2$ and $a_1 = 9$.

$$t^n = 6t^{n-1} - 9t^{n-2}$$

$$t^n - 6t^{n-1} + 9t^{n-2} = 0$$

$$t^{n-2}(t^2 - 6t + 9) = 0$$

$$(t-3)^2 = 0$$

$$t = 3$$

$$a_n = b \cdot 3^n + dn \cdot 3^n$$

$$a_0 = 2 = b + d \cdot 0 = b$$

$$a_1 = 9 = 2 \cdot 3 + d \cdot 3$$

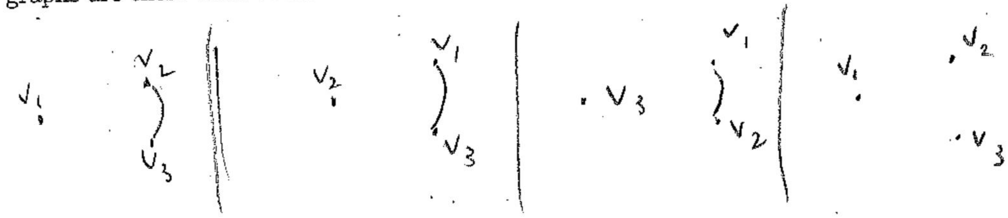
$$9 = 6 + 3d$$

$$3d = 3$$

$$d = 1$$

$$\boxed{a_n = 2 \cdot 3^n + n \cdot 3^n}$$

Exercise 0.7. Suppose $G = (V, E)$ is a graph. Say that a vertex $v \in V$ is *unfriendly* if it is connected by an edge to no other vertex. If $|V| = 3$, how many possible graphs are there with vertex set V that contain an unfriendly vertex?



4 graphs