Name: [redacted]

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MATH 61 - MIDTERM EXAM 1

0.1. **Instructions.** This is a 50 minute exam. You should feel free to quote any theorems proved in class, as well as anything proved in the homework or discussion section. There are 7 questions—on the real exam, you are required to do the first true/false question, and choose 5 of the remaining 6. Only 5 problems other than the true/false question will be graded so *you should indicate which problems you want graded by marking the one you do not want graded with an X*, in the case that you attempt all 6. Each question is worth 10 points. Unless otherwise specified, you are required to justify your answers.

Exercise 0.1. Indicate whether the following statements are true or false. You do not need to justify your answer.

(1) Suppose X and Y are finite sets with $|X|$ and $|Y|$ even. Then $|X \Delta Y|$ is even. True

(2) If $0 \leq k \leq n$, False

$$\binom{n}{k} = \binom{n-k}{k}. \quad \frac{n!}{(n-k)!k!} \overset{?}{=} \frac{(n-k)!}{\cancel{\ }k!k!}$$

(3) If $|X| = n$, there are n^n functions from X to X. False

(4) There are $2^n - 1$ sequences of 0s and 1s of length n with at least one 0. True

(5) Any sequence $\{a_n\}$ satisfying the recurrence relation

$$a_{n+2} = 4a_{n+1} + 2a_n$$

must have only even terms.

$a_0 = 0 \quad a_1 = 1 \quad a_2 = 4 \quad a_3 = 18$

$a_4 =$

as long as 0 is not odd (which it isn't)

① True

② False

③ False

④ True

⑤ False

$\underset{\text{total}}{2^n} - \underset{\text{no zeros}}{1}$

Initial conditions may be odd

0 0

$$\frac{n!}{(n-k)!\,k!} \overset{?}{=} \frac{(n-k)!}{(n-k-k)!\,k!}$$

$$\binom{5}{2} \overset{?}{=} \binom{3}{2} \text{ no}$$

Date: November 8, 2018; Ramsey.

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Don't Grade

Exercise 0.2. Suppose X and Y are finite sets with $n = |X| < |Y| = m$.

(1) Show there are $n!$ functions from X to X that are both injective and surjective.
(2) How many injective functions are there from X to Y?

Exercise 0.3. How many words can be obtained by rearranging the letters in

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(**Note:** they do not need to be real words and you can leave your answer in combinatorial notation)?

$$\frac{10!}{2! \cdot 2!}$$

Exercise 0.4. Show that if $2 \leq k \leq n$, then

$$\binom{n}{k} + 2\binom{n}{k-1} + \binom{n}{k-2} = \binom{n+2}{k}.$$

$$\binom{n}{k} + 2\binom{n}{k-1} + \binom{n}{k-2} = \binom{n}{k} + \binom{n}{k-1} + \underbrace{\binom{n}{k-1}\binom{n}{k-2}}$$

by theorem $= \underbrace{\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}}$

$$= \underbrace{\binom{n+1}{k}} + \underbrace{\binom{n+1}{k-1}}$$

again, by the same theorem

$$= \boxed{\binom{n+2}{k}}$$ QED

prove by pascal's triangle or combinatorially

Exercise 0.5. Let's pretend that there are no leap years so every year has 365 days. Show that if there are 185 students in our class, then there are two students who have either the same birthday or have consecutive birthdays.

b_T let set of birthdays $= \{b_1 \dots b_{185}\}$ $|b_T = 185|$

b_c set of consecutive birthdays $= \{b_1+1 \dots b_{185}+1\}$ $|b_c = 185|$

b_s set of same birthdays $= \{b_1^{+0} \dots b_{185}^{+0}\}$ $|b_s| = 185$

So $|b_s + b_c| = 185 + 185 = 370$, so there has to be 370 days for each student to have unique, non consecutive birthday. But, there are only 365 days.

By pigeonhole theorem, $|b_s| + |b_c| > 365$, so there are always two students who have either the same birthday or have consecutive birthdays. QED.

In other words, 370 one students consecutive day or a same day must be on the same day as another students.

Exercise 0.6. Find the solution to the recurrence relation

$$a_n = 6a_{n-1} - 9a_{n-2},$$

subject to the initial conditions $a_0 = 2$ and $a_1 = 9$.

$C_1 = 6 \quad C_2 = -9$

$t^2 - 6t + 9$

$(t-3)(t-3)$

$r_1 = r_2 = 3$

$a_n = b \cdot 3^n + d n 3^n$

$2 = b \quad a_0 = b \cdot 3^0 + d \cdot 0 3^0$

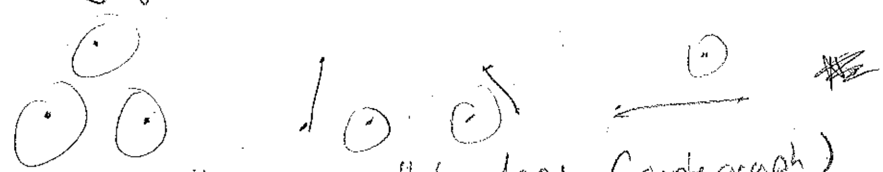
$9 = 2 \cdot 3 + d(1) \cdot 3$

$9 = 6 + 3d \quad d = 1$

$\boxed{a_n = 2 \cdot 3^n + n 3^n}$

Exercise 0.7. Suppose $G = (V, E)$ is a graph. Say that a vertex $v \in V$ is *unfriendly* if it is connected by an edge to no other vertex. If $|V| = 3$, how many possible graphs are there with vertex set V that contain an unfriendly vertex?

How many graphs with vertex set V, $|V| = 3$



assuming no loops or parallel edges. (simple graph)

$\boxed{4}$

There are no graphs w/ 2 unfriendly vertices b/c $|V| = 3$