

61 Midterm 2

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TOTAL POINTS

57 / 70

QUESTION 1

1 Exercise 1 10 / 10

- 2 pts (1) Incorrect [correct response T]
- 2 pts (2) Incorrect [correct response F]
- 2 pts (3) Incorrect [correct response T]
- 2 pts (4) Incorrect [correct response T]
- 2 pts (5) Incorrect [correct response F]

✓ - 0 pts All correct

QUESTION 2

2 Exercise 2 10 / 10

- 5 pts (1) Incorrect
- 1 pts (2) Incorrect: off-by-one error.
- 2 pts (2) Incorrect: gave answer m choose n instead of $(m \text{ choose } n) * n!$.
- 5 pts (2) Incorrect: other reason

✓ - 0 pts Both correct

- 10 pts Skipped

QUESTION 3

3 Exercise 3 10 / 10

- ✓ - 0 pts Correct
- 3 pts Miscalculated letters/repeats (e.g. answered $10!/2!$)
- 10 pts Not graded
- 8 pts Only recognized $10!$ permutations without considering repeats. (Or dealt with repeats incorrectly.)

QUESTION 4

4 Exercise 4 10 / 10

- ✓ - 0 pts Correct
- 10 pts Skipped
- 7 pts No further than expanding binomials (correctly)

- 2 pts Small algebra error
- 4 pts Made progress, but substantial algebra error or didn't finish
- 8 pts Showed some indication of what the individual terms mean combinatorially, but not how they're related
- 0 pts Click here to replace this description.

QUESTION 5

5 Exercise 5 7 / 10

- 0 pts Correct
- 10 pts Incorrect
- ✓ - 3 pts Apply Pigeonhole Incorrectly
- 8 pts Didn't apply Pigeonhole
- 10 pts Not Graded
- 5 pts Flawed argument, did not consider birthdays on same day

QUESTION 6

6 Exercise 6 10 / 10

- ✓ - 0 pts Correct
- 10 pts skipped
- 3 pts Algebra mistakes in solving for coefficients
- 5 pts Incorrect auxiliary polynomial
- 9 pts Incorrect, no attempt at application of method
- 6 pts No solution after finding aux polynomial and root
- 5 pts Incorrect solution form
- 5 pts Incorrect root for aux poly

QUESTION 7

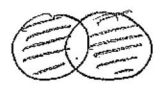
7 Exercise 7 0 / 10

- 0 pts Correct
- ✓ - 10 pts Skipped
- 1 pts Correct answer, but only justified by a picture

- **1 pts** Correct answer, but missing justification
- **4 pts** Omitted empty graph
- **9 pts** Incorrect, with no clear justification
- **8 pts** Miscalculated because had edges not connected to the given vertices
- **8 pts** Miscalculated because counted vertices that were connected to other vertices
- **5 pts** Did not include the empty graph in the count

MATH 61 - MIDTERM EXAM 1

0.1. Instructions. This is a 50 minute exam. You should feel free to quote any theorems proved in class, as well as anything proved in the homework or discussion section. There are 7 questions—on the real exam, you are required to do the first true/false question, and choose 5 of the remaining 6. Only 5 problems other than the true/false question will be graded so you should indicate which problems you want graded by marking the one you do not want graded with an X, in the case that you attempt all 6. Each question is worth 10 points. Unless otherwise specified, you are required to justify your answers.



Exercise 0.1. Indicate whether the following statements are true or false. You do not need to justify your answer.

$$|X-Y| + |Y-X| + |X \cap Y|$$

$$= |X| + |Y| - |X \cap Y|$$

$$|X-Y| + |Y-X| = |X \setminus Y| + |Y \setminus X|$$

$$= \text{even}$$

(1) Suppose X and Y are finite sets with $|X|$ and $|Y|$ even. Then $(X \Delta Y)$ is

True

(2) If $0 \leq k \leq n$,

$$\binom{n}{k} = \binom{n-k}{k} \cdot \binom{6}{3} \neq \binom{6-3}{3}$$

False

(3) If $|X| = n$, there are n^n functions from X to X .

True

(4) There are $2^n - 1$ sequences of 0s and 1s of length n with at least one 0.

True

(5) Any sequence $\{a_n\}$ satisfying the recurrence relation

$$a_{n+2} = 4a_{n+1} + 2a_n$$

must have only even terms. $= 2(2a_{n+1} + a_n)$ even

~~True~~ False

n steps,
each step
 n choices.

total all 1

$$= 2^n - 1$$

$$t^2 - 4t - 2$$

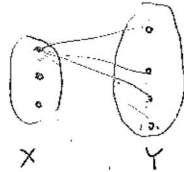
$$r_1 = \frac{b \pm \sqrt{b^2 - 4ac}}{2}$$

$$= \frac{4 \pm \sqrt{16 + 8}}{2} \quad 24 \quad 2\sqrt{6}$$

$$= 2 \pm \sqrt{6}$$

Exercise 0.2. Suppose X and Y are finite sets with $n = |X| < |Y| = m$.

- (1) Show there are $n!$ functions from X to X that are both injective and surjective.
- (2) How many injective functions are there from X to Y ?



(n options since $\text{range}(f) = X$)

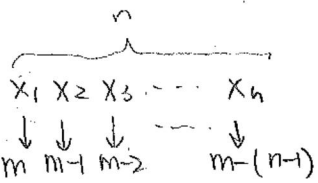
(1) $f: X \rightarrow X$. $|X| = n$. For $x_1 \in X$, there are n options in X such that $f(x_1) = \text{some } x \in X$. Since f is injective, for $x_2 \in X$, there are $(n-1)$ options left for x_2 s.t. $f(x_2) = \text{some } x \in X$. Similarly $x_n \in X$, there is 1 option left for x_n s.t. $f(x_n) = x \in X$. There are in total $n(n-1)\dots 1 = n!$ functions.

(2) For an injective $f: X \rightarrow Y$, distinct $x_1, x_2 \in X$ don't go to same y . The choices are $m(m-1)(m-2)\dots(m-n+1) =$

$$P(m, n) = \frac{m!}{(m-n)!}$$

r -perm

$$P(m, n) = \frac{m!}{(m-n)!}$$



Exercise 0.3. How many words can be obtained by rearranging the letters in

(CALIFORNIA)

(Note: they do not need to be real words and you can leave your answer in combinatorial notation?)

A: 2 I: 2

$$\frac{10!}{2!2!}$$

Exercise 0.4. Show that if $2 \leq k \leq n$, then

$$\binom{n}{k} + 2\binom{n}{k-1} + \binom{n}{k-2} = \binom{n+2}{k}$$

$$\frac{n!}{(n-k)!k!} + 2\frac{n!}{(n-k+1)!(k-1)!} + \frac{n!}{(n-k+2)!(k-2)!}$$

$$= \frac{n!(n-k+1)(n-k+2) + 2n!(n-k+2)k + n!k(k-1)}{(n-k+2)!k!}$$

$$= \frac{n!(n-k+2)(n-k+1+k) + n!k(n-k+2+k-1)}{(n-k+2)!k!}$$

$$= \frac{n![(n-k+2)(n+1) + k(n+1)]}{(n+2-k)!k!}$$

$$= \frac{n!(n+2)(n+1)}{(n+2-k)!k!}$$

$$= \frac{(n+2)!}{(n+2-k)!k!}$$

$$= \binom{n+2}{k}$$

Exercise 0.5. Let's pretend that there are no leap years so every year has 365 days. Show that if there are 185 students in our class, then there are two students who have either the same birthday or have consecutive birthdays.

pigeons: 365 days $365 > 185$
 holes: 185 students

We want to fit 185 students in 365 days, since $185 < 365$, there is at least one day that is the birthday of 2 students. (pigeon hole)

birthday $X: \{a_1, a_2, \dots, a_{185}\} \quad |X| = 185$

birthday+1 $X': \{a_1+1, a_2+1, \dots, a_{185}+1\} \quad |X'| = 185$

$|X| + |X'| = 370$ but we only have 365 days in a year.

Therefore $|X \cap X'| \neq \emptyset$. There is $x \in X$ and $x \in X'$ which means $a_j = a_i + 1$ for 2 birthdays.

$a_1, a_2, \dots, a_i, \dots, a_{185}$
 $a_1+1, \dots, a_j, \dots, a_{185}+1$

Exercise 0.6. Find the solution to the recurrence relation

$$a_n = 6a_{n-1} - 9a_{n-2},$$

subject to the initial conditions $a_0 = 2$ and $a_1 = 9$.

$$t^2 - 6t + 9 = 0$$

$$(t-3)^2 = 0$$

$$t = 3$$

$$a_0 = b + (0)d = b = 2$$

$$a_1 = 3b + 1 \cdot 3d = 9$$

$$b = 2, d = 1$$

$$a_n = 2 \cdot 3^n + n \cdot 3^n \quad \text{for } n \geq 2$$

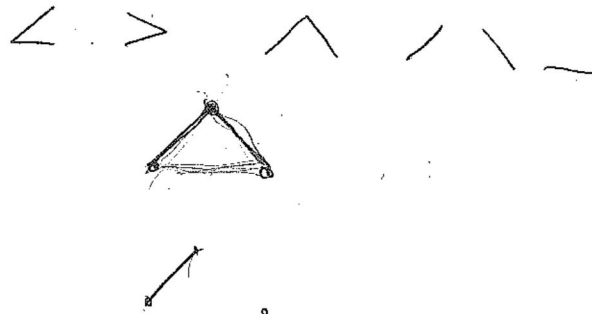
Exercise 0.7. Suppose $G = (V, E)$ is a graph. Say that a vertex $v \in V$ is unfriendly if it is connected by an edge to no other vertex. If $|V| = 3$, how many possible graphs are there with vertex set V that contain an unfriendly vertex?

	a	b	c
a	x	⊖	
b	⊕	x	⊖
c	⊖		x

$\delta(v) = 1$

contain 1:

contain 2:



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