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MATH 61 - MIDTERM EXAM 1

0.1. **Instructions.** This is a 50 minute exam. You should feel free to quote any theorems proved in class, as well as anything proved in the homework or discussion section. There are 7 questions—on the real exam, you are required to do the first true/false question, and choose 5 of the remaining 6. Only 5 problems other than the true/false question will be graded so *you should indicate which problems you want graded by marking the one you do not want graded with an X,* in the case that you attempt all 6. Each question is worth 10 points. Unless otherwise specified, you are required to justify your answers.

Exercise 0.1. Indicate whether the following statements are true or false. You do not need to justify your answer.

(1) Suppose X and Y are finite sets with $|X|$ and $|Y|$ even. Then $|X \Delta Y|$ is even.

(2) If $0 \leq k \leq n$,

$$\binom{n}{k} = \binom{n-k}{k}.$$

(3) If $|X| = n$, there are n^n functions from X to X.

(4) There are $2^n - 1$ sequences of 0s and 1s of length n with at least one 0.

(5) Any sequence $\{a_n\}$ satisfying the recurrence relation

$$a_{n+2} = 4a_{n+1} + 2a_n$$

must have only even terms.

$|X\Delta Y| = |X \cup Y| - |X \cap Y|$ No

$\binom{3}{2} \neq \binom{3}{2}$

n choices for each

No 0s is 1, total: 2^n => $2^n - 1$

$a_0 = 1$ aha :)

(1) T
(2) F
(3) T
(4) T
(5) F

$\{5,4\}$ $\{1,3\}$

Let $|X \cap Y| = x$

Then $|X \cup Y| = |X| + |Y| - |X \cap Y|$

$|X \Delta Y| = |X| + |Y| - 2|X \cap Y|$

=> even

Date: November 8, 2018; Ramsey.

Exercise 0.2. Suppose X and Y are finite sets with $n = |X| < |Y| = m$.

(1) Show there are $n!$ functions from X to X that are both injective and surjective.
(2) How many injective functions are there from X to Y?

(1) $|X| = n$

To be injective, no two elements in the domain may map to same element of codomain.

To be surjective, all of codomain is mapped to.

Since $|\text{domain}| = |\text{codomain}| = n$, injective is equivalent to surjective. each elements maps to distinct element. → all of codomain is in range

⇒ Number of injective functions is.

$n \cdot (n-1) \cdot (n-2) \cdots 1$

↑ n choices in codomain
↑ cannot repeat, so n-1
↑ same logic

These are all surjective.

⇒ $n!$ Bijective functions from X to X □

(2) $|Y| = m$ $|X| = n$

m choices in codomain for first element
m-1 for 2nd since it is injective, no repeats
m-2, so on...
⋮
m-n+1

n total

$\Rightarrow \prod_{i=0}^{n-1} (m-i) \Rightarrow \boxed{\frac{m!}{(m-n)!}}$ inj. functions from X to Y.

Exercise 0.3. How many words can be obtained by rearranging the letters in

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(**Note:** they do not need to be real words and you can leave your answer in combinatorial notation)?

10 letters — $10!$ ways of arranging if all distinct

2 As

2 Is — $2!$ ways of arranging 2 As and 2 Is

$$\Rightarrow \boxed{\frac{10!}{2!\,2!}}$$

$$= \frac{3628800}{4}$$

$$= \boxed{907200}$$

Exercise 0.4. Show that if $2 \le k \le n$, then

$$\binom{n}{k} + 2\binom{n}{k-1} + \binom{n}{k-2} = \binom{n+2}{k}.$$

$$\frac{n!}{k!(n-k)!} + \frac{n!}{(k-1)!(n-k+1)!} = \frac{n!}{(k-1)!(n-k)!}\left(\frac{1}{k} + \frac{1}{n-k+1}\right)$$

$$= \frac{n!}{(k-1)!(n-k)!}\left(\frac{n+1}{(k)(n-k+1)}\right) = \frac{(n+1)!}{(k)!(n-k+1)!} = \binom{n+1}{k}$$

⇑ Proof

Algebraic:

~~Algebra:~~

~~$\frac{n!}{k!(n-k)!} + \frac{2n!}{(k-1)!(n-k+1)!} + \frac{n!}{}$~~

Since we know $\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$, (Pascal's Identity) (*)

$\binom{n}{k} + 2\binom{n}{k-1} + \binom{n}{k-2} \overset{?}{=}$

$\Leftrightarrow \binom{n}{k} + \binom{n}{k-1} + \binom{n}{k-1} + \binom{n}{k-2}$

Applying (*)

$\Leftrightarrow \binom{n+1}{k} + \binom{n+1}{k-1}$

Applying (*)

$\Leftrightarrow \binom{n+2}{k}$, the desired answer.

All steps are reversible so we are done. □

Combinatorial:

(n+2) things

(n)

2 special ones

Ways of picking k things from n+2 total $= \binom{n+2}{k}$

Case 1: Pick from group of n: $\binom{n}{k}$

Case 2: Pick 1 from special, k-1 from the n

$\Rightarrow \binom{2}{1} \cdot \binom{n}{k}$

Case 3: Pick 2 from special, k-2 from the n.

$\Rightarrow \binom{2}{2}\binom{n}{k-2}$

⇒ These are equivalent so we are done. □

Exercise 0.5. Let's pretend that there are no leap years so every year has 365 days. Show that if there are 185 students in our class, then there are two students who have either the same birthday or have consecutive birthdays.

Let b_i from $i = 0$ to $i = 184$ be the ~~[illegible]~~, where $0 \le b_i < 365$ birthdays of the people

Each day is given what day it is in the year.

Then let $b_i' = b_i + 1$ from $i = 0$ to 184

$1 \le b_i' < 366$

~~Then [illegible]~~

If $b_i = b_j$ where $i \neq j$, two students have the same birthday, and we are done.

Now, let $b_i \neq b_j$ for all $i \neq j$.

Then b_i are all distinct and b_i' are all distinct (since b_i' is a bijection of b_i)

However, the range of $\{b_i, b_i'\}$ is from 0 to 355.

There are 185 b_i and 185 b_i' => 370 total, only 366 holes.

So by pigeonhole, there must be some $b_i = b_j'$ since all b_i are distinct and b_j' are distinct.

=> There are two people whose birthdays differ by 1.

Thus, we have proved the desired. □

Exercise 0.6. Find the solution to the recurrence relation

$$a_n = 6a_{n-1} - 9a_{n-2},$$

subject to the initial conditions $a_0 = 2$ and $a_1 = 9$.

$a_n = 6a_{n-1} - 9a_{n-2}$

$\Leftrightarrow a_n - 6a_{n-1} + 9a_{n-2}$

Polynomial: $t^2 - 6t + 9 = 0$

$t = 3$, repeated

$\Leftrightarrow a_n = b(3)^n + d \cdot n(3)^n$

$a_0 = b + 0 = 2 \Leftrightarrow b = 2$

$a_1 = 3b + 3d = 9 \Leftrightarrow b + 3d = 9$

$b = 2 \Leftrightarrow d = 1$

$\Updownarrow$

$a_n = 2(3)^n + n(3)^n = \boxed{(3)^n(2+n)}$ D.

Check:

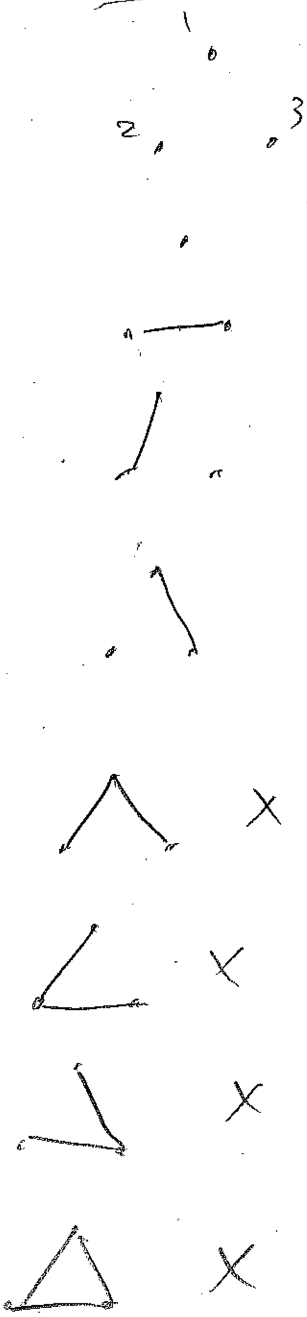
$(3^n)(2+n) = 6(3^{n-1})(2+n-1) - 9(3)^{n-2}(2+n-2)$

$9(2+n) = 18(1+n) - 9n$

$18 + 9n = 18 + 18n - 9n$

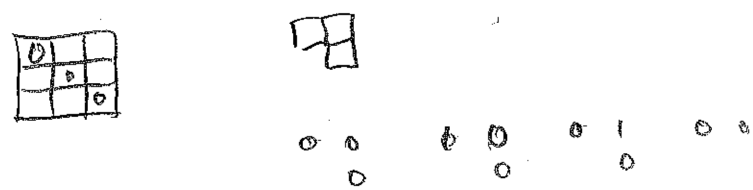
$18 + 9n = 18 + 9n$ ✓ D.

Exercise 0.7. Suppose $G = (V, E)$ is a graph. Say that a vertex $v \in V$ is *unfriendly* if it is connected by an edge to no other vertex. If $|V| = 3$, how many possible graphs are there with vertex set V that contain an unfriendly vertex?



If no direction: 4

~~If directed: 7~~



=> 4