61 Midterm 2

TOTAL POINTS

54/60

QUESTION 1

1 Exercise 1 6 / 10

- **2 pts** (1) Incorrect [correct response T]
✓ - **2 pts (2) Incorrect [correct response F]**
✓ - **2 pts (3) Incorrect [correct response T]**
- **2 pts** (4) Incorrect [correct response T]
- **2 pts** (5) Incorrect [correct response F]
- **0 pts** All correct

(2) If n = k > 0, then the left hand side is positive and the right hand side is either 0 or undefined.
(3) Each of n elements in X has n choices for where it goes.

QUESTION 2

2 Exercise 2 0 / 10

- **5 pts** (1) Incorrect
- **1 pts** (2) Incorrect: off-by-one error.
- **2 pts** (2) Incorrect: gave answer m choose n instead of (m choose n) * n!.
- **5 pts** (2) Incorrect: other reason
- **0 pts** Both correct
✓ - **10 pts Skipped**

QUESTION 3

3 Exercise 3 10 / 10

✓ - **0 pts Correct**
- **3 pts** Miscounted letters/repeats (e.g. answered 10!/2!)
- **10 pts** Not graded
- **8 pts** Only recognized 10! permutations without considering repeats. (Or dealt with repeats incorrectly.)

QUESTION 4

4 Exercise 4 10 / 10

✓ - **0 pts Correct**
- **10 pts** Skipped
- **7 pts** No further than expanding binomials (correctly)
- **2 pts** Small algebra error
- **4 pts** Made progress, but substantial algebra error or didn't finish
- **8 pts** Showed some indication of what the individual terms mean combinatorially, but not how they're related
- **0 pts** Click here to replace this description.

QUESTION 5

5 Exercise 5 8 / 10

- **0 pts** Correct
- **10 pts** Incorrect
- **3 pts** Apply Pigeonhole Incorrectly
- **8 pts** Didn't apply Pigeonhole
- **10 pts** Not Graded
- **5 pts** Flawed argument, did not consider birthdays on same day

- 2 Point adjustment

must explicitly assume distinct birthdays

QUESTION 6

6 Exercise 6 10 / 10

✓ - **0 pts Correct**
- **10 pts** skipped
- **3 pts** Algebra mistakes in solving for coefficients
- **5 pts** Incorrect auxiliary polynomial
- **9 pts** Incorrect, no attempt at application of method
- **6 pts** No solution after finding aux polynomial and root
- **5 pts** Incorrect solution form

- **- 5 pts** Incorrect root for aux poly

QUESTION 7

7 Exercise 7 **10 / 10**

- ✓ **- 0 pts** **Correct**
- **- 10 pts** Skipped
- **- 1 pts** Correct answer, but only justified by a picture
- **- 1 pts** Correct answer, but missing justification
- **- 4 pts** Omitted empty graph
- **- 9 pts** Incorrect, with no clear justification
- **- 8 pts** Miscounted because had edges not connected to the given vertices
- **- 8 pts** Miscounted because counted vertices that were connected to other vertices
- **- 5 pts** Did not include the empty graph in the count

Grade 3, 4, 5, 6, 7

MATH 61 - MIDTERM EXAM 1

0.1. **Instructions.** This is a 50 minute exam. You should feel free to quote any theorems proved in class, as well as anything proved in the homework or discussion section. There are 7 questions—on the real exam, you are required to do the first true/false question, and choose 5 of the remaining 6. Only 5 problems other than the true/false question will be graded so *you should indicate which problems you want graded by marking the one you do not want graded with an X*, in the case that you attempt all 6. Each question is worth 10 points. Unless otherwise specified, you are required to justify your answers.

Exercise 0.1. Indicate whether the following statements are true or false. You do not need to justify your answer.

(1) Suppose X and Y are finite sets with $|X|$ and $|Y|$ even. Then $|X \Delta Y|$ is even.

(2) If $0 \leq k \leq n$,
$$\binom{n}{k} = \binom{n-k}{k}.$$

(3) If $|X| = n$, there are n^n functions from X to X.

(4) There are $2^n - 1$ sequences of 0s and 1s of length n with at least one 0.

(5) Any sequence $\{a_n\}$ satisfying the recurrence relation
$$a_{n+2} = 4a_{n+1} + 2a_n$$
must have only even terms.

$\cup - \cap$

1356 1457

13456715

1356, 1489

1345689

(1) True

(2) True

(3) False

(4) True

(5) False

$\frac{n!}{n-k!\,k!}$ $\frac{n-k!}{(n-k-k)!\,k!}$

0 _ _ _

$2^n - 1$

1 2 3 n

(1,1)

9 2^{n^2}

Date: November 8, 2018; Ramsey.

Exercise 0.2. Suppose X and Y are finite sets with $n = |X| < |Y| = m$.

(1) Show there are $n!$ functions from X to X that are both injective and surjective.
one-to-one onto

(2) How many injective functions are there from X to Y?

Exercise 0.3. How many words can be obtained by rearranging the letters in

$$CALIFORNIA$$

(**Note:** they do not need to be real words and you can leave your answer in combinatorial notation)?

$$\boxed{\frac{10!}{\underset{A}{2!}\,\underset{I}{2!}}}$$

C 1
A 2
L 1
I 2
F 1
O 1
R 1
N 1

$$\frac{(n+2)!}{(n-k+2)!\,k!}$$

Exercise 0.4. Show that if $2 \le k \le n$, then
$$\binom{n}{k} + 2\binom{n}{k-1} + \binom{n}{k-2} = \binom{n+2}{k}. \longrightarrow$$

LHS:

$$= \frac{n!}{(n-k)!\,k!} + 2\frac{n!}{(n-k+1)!(k-1)!} + \frac{n!}{(n-k+2)!(k-2)!} =$$

$$\frac{(n-k+2)(n-k+1)}{(n-k+2)(n-k+1)}\left(\frac{n!}{(n-k)!\,k!}\right) + 2\frac{n!}{(n-k+1)!(k-1)!}\left(\frac{(n-k+2)k}{(n-k+2)k}\right) + \left(\frac{n!}{(n-k+2)!(k-2)!}\right)\left(\frac{(k-1)k}{(k-1)k}\right)$$

numerator:

$$(n-k+2)(n-k+1) = n^2 - nk - nk + n + k^2 - k + 2n - 2k + 2$$
$$= n^2 + 3n + 2 - 2nk + k^2 - 3k$$

$$2(n-k+2)(k) = 2nk - 2k^2 + 4k$$

$$(k-1)(k) = k^2 - k$$

$$\frac{n!(n^2+3n+2 \cancel{-2nk} \cancel{+k^2} \cancel{-3k} \cancel{+2nk} \cancel{-2k^2} \cancel{+4k} \cancel{+k^2} \cancel{-k})}{(n-k+2)!\,k!}$$

$$= \frac{n!(n^2+3n+2)}{(n-k+2)!\,k!} = \frac{n!(n+2)(n+1)}{(n-k+2)!\,k!} = \frac{(n+2)!}{(n-k+2)!\,k!} = \binom{n+2}{k} \checkmark$$

Exercise 0.5. Let's pretend that there are no leap years so every year has 365 days. Show that if there are 185 students in our class, then there are two students who have either the same birthday or have consecutive birthdays.

$a_1 \cdots a_{185}$

$a_1+1 \cdots a_{185}+1$

We have 370 (terms) numbers, which range in value from 1 to 366. Therefore, if the terms are pigeons and the values are pigeon holes, we must have one pigeonhole with at least two pigeons. As such, this means that

$a_i = a_j + 1$ for some $i \neq j$

So there are two students who have either the same birthday or have consecutive birthdays.

Exercise 0.6. Find the solution to the recurrence relation
$$a_n = 6a_{n-1} - 9a_{n-2},$$
subject to the initial conditions $a_0 = 2$ and $a_1 = 9$.

$t^2 - 6t + 9 = 0$

$(t-3)^2 = 0$

$t = 3$ repeated root

$\Rightarrow \quad a_n = b(3)^n + dn(3)^n$

Given $a_0 = 2$

$2 = b(3)^0 + d(0)(3)^0$

$\Rightarrow b = 2$

Given $a_1 = 9$

$9 = b(3)^1 + d(3)^1$

$9 = 3b + 3d$

$\Rightarrow 9 = 6 + 3d$

$3 = 3d$

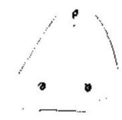
$d = 1$

$\therefore \boxed{a_n = 2(3)^n + n(3)^n}$

Exercise 0.7. Suppose $G = (V, E)$ is a graph. Say that a vertex $v \in V$ is *unfriendly* if it is connected by an edge to no other vertex. If $|V| = 3$, how many possible graphs are there with vertex set V that contain an unfriendly vertex?

$|V| = 3$

• ← unfriendly



graphs that contain unfriendly vertex = # graphs with 0 or 1 edges

0 edge graphs = 1

3 possible edges

1 edge graphs = $\binom{3}{1} = 3$

select 1 edge

Union of these two = $3 + 1 = 4$

$\boxed{\text{4 graphs with an unfrienly vertex}}$